The Iterative Deferred Acceptance Mechanism[*]

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November 2017

Abstract

We introduce a new mechanism for one-sided matching markets, inspired by procedures currently being used to match millions of students to public universities in Brazil and China. Unlike most mechanisms available in the literature, which ask students for a full preference ranking over all colleges, they are instead sequentially asked to make choices among sets of colleges. These choices are used to produce, in each step, a tentative allocation. If at some point it is determined that a student cannot be accepted into a college, then she is asked to make another choice among those which would tentatively accept her. Participants following the simple strategy of choosing the most preferred college in each step is a robust equilibrium that yields the Student Optimal Stable Matching. We also provide an extension in which, after running the sequential mechanism for a number of steps students are asked to submit a ranking over the colleges that are still within reach. This constitutes a novel approach to matching mechanisms. We show that the initial sequential stage clears a substantial part of the market before the rankings submission. This finding, together with empirical and simulation results, makes our proposal an attractive alternative to the sequential mechanisms currently being used and the standard Gale-Shapley Deferred Acceptance mechanism for practical applications.

JEL classification: C78, C73, D78, D82

Keywords: Market Design, Matching, Sequential Mechanisms, College Admissions.

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1   Introduction

The field of market design has developed rapidly in recent years, both in terms of the range of objectives that are studied – different notions of efficiency, stability, fairness, etc. – and also in the number of applications and their evaluations, in the field and in the lab. In the typical framework, the attainability of a given objective is evaluated in terms of mechanisms that request the relevant agents to submit preferences over sets of outcomes, before a clearinghouse then combines them using some predetermined criteria to produce an allocation. This induces a game in which the action space of the participants consists of the preferences they submit. By studying the incentive properties of these games, one can then see how equilibrium outcomes relate to the objectives of the market designer. These mechanisms, therefore, have two common properties: they are direct (in the sense that the participants are asked for their relevant types, in this case their preferences,) and induce a simultaneous move game: all agents simultaneously interact only once.

There are many theoretical and practical reasons for using and focusing on direct mechanisms. First, the revelation principle guarantees that nothing is lost by using direct mechanisms as opposed to alternative action spaces. Second, in the induced games the participants have a simple strategy space, whereas strategies in sequential games may consist of large sets of contingency plans over information structures. Finally, if truth-telling is a dominant strategy (i.e., the mechanism is strategy-proof), the game can be understood as being very simple for a participant, with truth-telling being the expected behavior.

In this paper we follow a different path and propose a family of sequential mechanisms for implementing stable outcomes in many-to-one matching markets. These are problems for which there is already a well-known strategy-proof direct mechanism, namely the Gale-Shapley student-proposing deferred acceptance procedure (DA). In the mechanisms we propose, instead of requesting from each participant a full preference over all private outcomes, they are instead repeatedly asked to make choices, which are used to produce tentative allocations. Some information about these allocations may be given back to the participants, before asking them for further choices, until a final allocation is produced. The mechanism, therefore, resembles a sequential implementation of DA, in which instead of using a preference ranking submitted by the agents to make the choices during the process, the agents themselves are repeatedly asked to make choices from sets of available options. We term them by iterative deferred acceptance mechanisms, and use two main reasons to justify this departure.

First, the last few years have seen the emergence of uses of iterative mechanisms for

matching students to schools and colleges, some of them on a very large scale. Prominent examples are the college admission mechanism for the Chinese province of Inner Mongolia [Chen and Pereyra, 2016, Gong and Liang, 2016], used for matching more than 200,000 students to universities per year, the SISU mechanism, used to determine where more than two million prospective university students per year are matched, among public universities in Brazil, and the mechanism currently being used in the German university admissions [Grenet et al., 2017].[1] The use of these procedures has been made possible, in practice, by the Internet, which allows students to easily interact multiple times with a central clearinghouse via a website or even a mobile application. Although the procedures being used improve upon many aspects of the ones previously in place, an evaluation of its theoretical and empirical properties is imperative, given the potential impact on large numbers of students.

Second, there are reasons why the standard DA mechanism may not have the desired theoretical properties and outcomes, especially in large-scale college admissions, where there are thousands of options available to the students. One is that students are, in practice, either restricted to list a limited number of options, or even if they could do so it is actually impractical to require students to rank all of those options in a way that truly represents their preferences. As shown in Haeringer and Klijn [2009], when lists are restricted, truth-telling is no longer a dominant strategy, and equilibria may not be stable. That is, the two main benefits of DA are eliminated. Another one is that there is empirical and experimental evidence that the strategic simplicity of DA may not be matched by an understanding, on the part of the agents, of its incentives [Chen and Sönmez, 2006, Pais and Pintér, 2008, Ding and Schotter, 2017, Rees-Jones, forthcoming, Hassidim et al., 2015, Chen and Pereyra, 2015]. Finally, recent experimental evidence shows that iterative versions of DA outperform the standard one in the proportion of stable outcomes and truth-telling rates [Klijn et al., 2016, Bo and Hakimov, 2016].

This paper is divided into three parts. In section 2 we analyze the mechanism currently being used to match students to public universities in Brazil, denoted SISU. Under the SISU mechanism, students are repeatedly asked to choose one college from the list of options available. At the end of each of four days, cutoff grades, which represent the tentative minimum requirements for acceptance at each college, are made public. Students are then allowed to change their choice in response to that information. After a predetermined number of steps, the last choices made are used to produce an allocation. We show that the SISU mechanism has some undesirable theoretical properties: it may fail to give reliable information about

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[1]Another example of current use of iterative mechanisms is the school district in Wake County, NC. [Dur et al., forthcoming]

where students could be accepted, and it is subject to a new type of manipulation, denoted manipulation via cutoffs. Using detailed data on the values of the cutoffs at the end of each day during the selection process that took place in 2016, we show that the first problem is empirically relevant, that is, that the cutoff values produced by it often constitute unreliable information for the students. Manipulation via cutoffs are situations in which groups of students with high exam grades temporarily inflate the cutoff grades at some colleges and change their options in the last step, with the objective of reducing the competition faced by specific low-grade students. We show that, due to specific characteristics of college admissions in these countries, these manipulations are feasible both in Brazil and in Inner Mongolia, and provide anecdotal evidence that they take place in real life in the latter.

In the second part, in section 3, we introduce the Generalized Iterative Deferred Acceptance Mechanism (GIDAM), as well as its special case for the exam-based college admissions, the Iterative Deferred Acceptance Mechanism (IDAM). Both of them are free from the problems identified above. Similarly to the SISU mechanism, under the GIDAM students are repeatedly asked for choices from sets of options until an allocation is produced. It differs, however, in other dimensions. In every step, it restricts students to only choosing colleges which would accept them given other students' choices in that step. Also, it introduces "commitment" in students' choices: students are not allowed to change their choices unless the previous ones are no longer feasible. Moreover, it allows students to submit partial rankings (as opposed to single colleges) in each step, and for different contractual terms between students and colleges (such as with and without funding). When students follow the simple strategy of choosing the most preferred college among those available at each step (denoted the *straightforward behavior*),[2] the matching produced after a finite number of steps is the Student Optimal Stable Matching, that is, the matching that is the most preferred by all students among all stable matchings (Proposition 1). Moreover, students following the straightforward behavior constitute a robust equilibrium in which any deviating strategy is stochastically dominated by following the equilibrium one (Theorem 1). This result comes from the fact that, with the modifications introduced with respect to the SISU, deviating strategies are indistinguishable, from the perspective of an observer, from the straightforward behavior for some different preferences.

In the third part, we show that the comparison between DA and IDAM presents a trade-off between the length of submitted preferences in DA and the number of steps that IDAM takes to produce its outcome. Section 4 evaluates this trade-off through theoretical and

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[2]As we will show later, the straightforward behavior includes students also submitting (possibly constrained) rankings over available colleges.

simulation results. We run simulations comparing the number of steps it takes for the IDAM mechanism to produce an outcome and the minimum length of a rank-ordered list necessary for truth-telling to be an equilibrium in DA. These show that the relative advantage of the IDAM is higher when the students to seats ratio is higher. Interestingly, when the number of students equals the number of seats, the simulations also show that IDAM produces an outcome in fewer steps in scenarios where DA needs longer rank ordered lists and vice-versa.

Finally, we present a new mechanism for matching students to colleges, which is a "hybrid" between iterative mechanisms and DA, denoted IDAM+DA. It consists of two parts. First, we run a pre-specified number of steps of IDAM. After that, we ask students to submit a ranking over those colleges still available to each of them after these initial steps, and then execute a deferred acceptance procedure over this "residual" problem. This being a special case of the GIDAM mechanism, truthful behavior is also a robust equilibrium, producing the student-optimal stable matching. That is, during k steps, students are asked to choose among all options available, and if rejected to make new choices among the remaining, still feasible, options. After these k steps, the minimum grade required for being accepted at some colleges are such that some students, especially those who have average or low exam grades, see a substantial reduction in the number of colleges where they may still be matched. Students are then requested to submit a preference ranking over the remaining options. By using a model with a continuum of students, we show that when students are ranked in the same way by all colleges these initial steps clear a significant proportion of the market, that is, a large number are matched to their final outcome with these steps (Proposition 5). Moreover, the marginal gain is higher in the first steps. This implies that by running a small number of steps of IDAM before the "residual DA" problem we clear the market for many students, and also as a result reduce the set of remaining options that students need to rank. These results are robust to more general settings, as is shown in simulations with more general settings. The IDAM+DA mechanism is, therefore, a new and appealing alternative for DA for a policymaker. It runs for a pre-specified number of steps, clears the market for a large proportion of students with a small number of choices, and greatly reduces the size of the ranking that is necessary to ask students to submit in the last step.

1.1   Related literature

This paper mainly relates to two lines of research in market design. One is the family of works which evaluate, from both a positive and a normative perspective, mechanisms that are being used in the field in college admission and school choice. While evaluating the col-

lege admission process in Turkey, Balinski and Sönmez [1999] showed that the Gale-Shapley student-proposing deferred acceptance procedure (DA) [Gale and Shapley, 1962] is characterized as the "best" fair mechanism, in that it is strategy-proof and Pareto dominates any other fair mechanism (that is, it is constrained efficient). In fact, variations of the DA mechanism are used in many real-life student matching programs around the world. Examples of the use of the DA mechanism include the college and secondary school admissions in Hungary [Biró, 2011], the high school admissions in Chicago [Pathak and Sönmez, 2013], the New York City [Abdulkadiroğlu et al., 2009] and the elementary schools in Boston [Abdulkadiroglu et al., 2006]. Other mechanisms, such as the college-proposing DA, top trading cycles, the so-called "Boston mechanism" and the "Parallell mechanism" are used to match millions of students to schools and colleges around the world [Chen and Kesten, 2017, Abdulkadiroğlu and Sönmez, 2003, Balinski and Sönmez, 1999]. Gong and Liang [2016] and Chen and Pereyra [2016] apply the mechanism currently in use to match students to universities in the province of Inner Mongolia in China. Although the dynamic mechanism used in that province has some similarities to the SISU, such as the availability of tentative cutoff grades, it is in fact a different mechanism, with different timing and incentives. Grenet et al. [2017] analyze the system used for college admissions in Germany, which combines a sequential phase and a direct revelation phase. In section 4.1 we show that the IDAM+DA mechanism also presents this combination and has good practical properties.

The other line in which this paper relates is that of the study of sequential mechanisms. Kagel et al. [1987] show that, although the second-price auction is isomorphic to an English auction, experiments show that behavior is significantly different when comparing both, with truthful behavior more prevalent in the latter. Ausubel [2004] and Ausubel [2006] propose sequential auction mechanisms for multiple (homogeneous and heterogeneous, respectively) objects. While there are direct mechanisms which implement the same outcomes in dominant strategies, the author argues that the proposed sequential mechanisms are simpler and preserve the privacy of participants.

In a recent paper, Li [2017] provides a theoretical justification for why some sequential mechanisms perform better than their direct counterparts. That justification is based on a refinement of strategy-proofness, denoted obvious strategy-proofness (OSP), in which the realization that a certain strategy is dominant does not rely on contingent reasoning. The author shows that a family of mechanisms, which includes the English auction, is OSP, therefore providing a theoretical explanation for the results in Kagel et al. [1987]. When it comes to stable mechanisms, however, Ashlagi and Gonczarowski [2015] show that there is

no OSP mechanism which yields stable matchings.[3]

Experimentally comparing the behavior under DA and the IDAM mechanism, Bo and Hakimov [2016] show that the truthful equilibrium in IDAM, which produces the student-optimal stable matching, predicts behavior better than the dominant strategy in DA, also leading to a larger proportion of stable outcomes. Klijn et al. [2016] also present evidence in that direction. Similarly, Kagel and Levin [2009] show experimental evidence that subjects behave more often in line with the equilibrium prediction in the sequential mechanisms in Ausubel [2004] than with the dominant strategy of the direct versions of it. These results indicate that the behavior more consistent with the equilibrium prediction in these sequential implementations is not entirely captured by the refinement proposed in Li [2017].

Other papers have evaluated non-direct iterative mechanisms for matching students to colleges or schools. Dur et al. [forthcoming] use the fact that the school choice mechanism used in the Wake County Public School System allows for students to interact multiple times with the procedure as a method for empirically identifying strategic players. Interestingly, the dynamic nature of the procedure, and the information that is made available during the process to the participants, makes it somewhat comparable to the IDAM mechanism. A rich series of papers also consider sequential mechanisms which implement stable matchings in equilibrium, including Alcalde and Romero-Medina [2000], Alcalde and Romero-Medina [2005], Romero-Medina and Triossi [2014], and Klaus and Klijn [2017]. While many of these mechanisms implement stable allocations in equilibrium, the determination of equilibrium strategies depends on coordination between students in a way that is significantly more demanding than the equilibrium strategy that IDAM has, which depends solely on (partial) information about the student's own preferences over colleges.

Proofs absent from the main text and additional details can be found in the appendix.

2 The SISU mechanism

In 2010 the Brazilian ministry of education launched a method for matching students to university programs, denoted SISU. The SISU system represented a significant change in the way in which universities admitted students. First, it unified the acceptance criteria at the universities for the seats made available through the system: instead of a different exam for each university, a unified national exam was used. Second, students were free to apply to any

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[3]The authors show, however, that when the preferences on one side of the market satisfy an acyclicity condition there is an OSP stable mechanism. This is a very restrictive condition, which is not satisfied, for example, by the college admission process in Brazil.

program in any university in the country (among those available in the SISU) for no extra cost, whereas previously in some cases the student would have to travel to the university premises just to be able to apply. Third, and perhaps most importantly, the centralized system could allow a student to obtain information about which university programs would accept her.

In the period between 2010 and 2016, the precise rules which define the SISU mechanism were changed multiple times. The version that we will consider for analysis, due to its simplicity, is the one used in the year 2010. Although later versions have different modifications, to the best of our knowledge all the problems identified in this section are also present in the later versions.

In the setup used in the SISU mechanism, there is a set of colleges $C = \{c_1, \ldots, c_m\}$ with fixed capacities (a maximum number of students who can be matched to them) $(q_{c_1}, \ldots, q_{c_m})$. Colleges rank all students based on the results of the national exam. Different colleges[4] may use different weights for the various parts of the exam. For example, economics programs could give a higher weight to the math section of the exam, while medical programs could give a higher weight to the biology section. Denote by $z_c(s)$ student s's resulting exam grade in college c. Colleges may also have a minimum acceptance grade, representing the minimum value of $z_c(s)$ a student s must have to be acceptable at c, denoted $\underline{z}_c$. Given a set of students applying to a certain college, a commonly used information is the *cutoff* grade for that college. A cutoff grade represents the lowest grade necessary to be accepted at a college, given the set of students applying to it. When looking at the cutoff values of all colleges, therefore, a student can infer which ones would accept her if all other students' choices remain constant. Before the SISU mechanism was introduced, it was common for students to see historical values of the cutoffs for the different colleges as an indication of where they should apply, given their information about their own exam grade or ability. One of the advantages of the new procedure would be to allow the students to make that assessment in "real-time" instead of only based on historical data.

The mechanism runs for four days.

- During each day $t = \{1, 2, 3, 4\}$, students may each choose a college to apply for. If a student makes no choice, her last choice is used again, if any. At the end of the each of the first three days, the following is executed, for each college c:

    - If the number of students who chose c and have an exam grade at that college

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[4]In Brazil, as in many other countries, students apply directly to specific programs in the colleges or universities. For simplicity, though, we refer only to "colleges" whenever the distinction is not necessary.

higher than $\underline{z}_c$ is smaller than q_c, the cutoff grade ζ_c^t is set to $\underline{z}_c$.

– Otherwise, the cutoff grade ζ_c^t is set to be the q_c^{th} highest grade at that college among those who chose it in that day.

- The values of $\zeta_{c_1}^t, \ldots, \zeta_{c_m}^t$ are made public.

- At the end of the fourth day, a matching of students to colleges is produced, as follows:

    – For each college c, the top q_c students who have an exam grade higher than $\underline{z}_c$ and chose c in the last day are matched to it.

    – All students who were not among the ones above remain unmatched.

    – Final cutoffs, calculated in the same way, are made public.

The definition above can be naturally extended to the case where the number of days is any arbitrary number T^{Max}, so we may also consider this more general definition as well. Although the potential ability to know which colleges a student might not be matched to before submitting their final choice seems like an interesting property, in fact that is not the case in general, as is noted in the following remarks.

Remark 1. Choices made in the run-up to the last day may have no direct effect on the final outcome. As a result, students have no clear incentive to make choices prior to the last day.

Of course, if some student makes a choice the day before the last and does not make a choice on the last, her last choice will be the one considered when generating the outcome. However, the outcome would be the same if we kept other players' choices and that student made her choice only on the last day. The fact that this results in no clear incentive for students to make choices before the last day makes the information available before the beginning of the last day, regarding which colleges are out of reach, even less reliable.

Remark 2. The cutoff values at some colleges may go down from one day to the next.

Since students may choose any college on any day, nothing prevents the cutoff values at some colleges from going down from one day to the next. For example, consider a scenario in which $T^{Max} = 4$ and college c has unit capacity. Let student s, where $z_c(s) = 200$, be the only student to choose college c during day three. The cutoff value for c made public at the end of day three is therefore $\zeta_c^3 = 200$. If s chooses a different college on day four and no other student chooses c, then $\zeta_c^4 = \underline{z}_c$. That is, some student s' whose grade at c is greater than $\underline{z}_c$ but lower than 200, cannot take the cutoff value at college c, even at the end of day three, as an indication that she had no chance of being accepted there by the end of day four.

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If the cutoff values go down from one day to another, then the use of those values as information that guides students' applications away from colleges at which they will not be accepted is jeopardized. Moreover, if the cutoff values go down at some program from day $T^{Max} - 1$ to T^{Max}, a student who may have preferred to go to that program and would be accepted by the end of day T^{Max} will not do so.

Another shortcoming of the SISU mechanism is that it is subject to a new type of manipulation, denoted manipulation via cutoffs, in which groups of students may induce others to change their behavior in a way that may benefit that group. This is explored in more detail in section 2.2.

2.1 Empirical evidence

In order to evaluate the empirical relevance of the shortcomings of the SISU mechanism identified in the previous section, we analyze data for the selection process that took place in January 2016. In that month, more than 228,000 seats in public universities were offered, and a total of more than 2,500,000 students participated. The average competition, therefore, was of more than 10 candidates per seat.

The data consists of the cutoff values for each of the 25,686 options available to the students, for each of the four days in which students were able to make choices.[5] In Brazilian universities, students apply and may be accepted to specific programs in those universities, as opposed to joining the university as a whole. For example, a student must choose whether to apply for the daytime economics program at the Federal University of Rio de Janeiro, or for the night-time computer science program at the same university. Although all programs use the same national university entrance exam, different programs may use different weighted averages for different parts of the exam when ranking students.

In the present analysis, we are interested in whether the cutoff values decrease from one day to another and, if so, by how much. As pointed out in section 2, a decrease in the cutoff values points to a failure of the SISU mechanism in providing information on the programs

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[5]The process in 2016 differs from the rules used in 2010 in two aspects. First, the seats in each program offered by federal public universities are split into up to five different sets of seats, and eligibility to apply to each of those sets of seats differ across students (see Aygün and Bó [2013] for more details on the procedure). Secondly, instead of choosing only one program per day, students were able to specify first and second choices. Each day, DA was run based on the two submitted choices.

Finally, we note that although many programs have minimum exam grades in order for a student to be acceptable at that program, some do not, so cutoff values are set to zero if the number of students acceptable who chose one of these programs is below the capacity. More information on these differences can be found in the appendix.
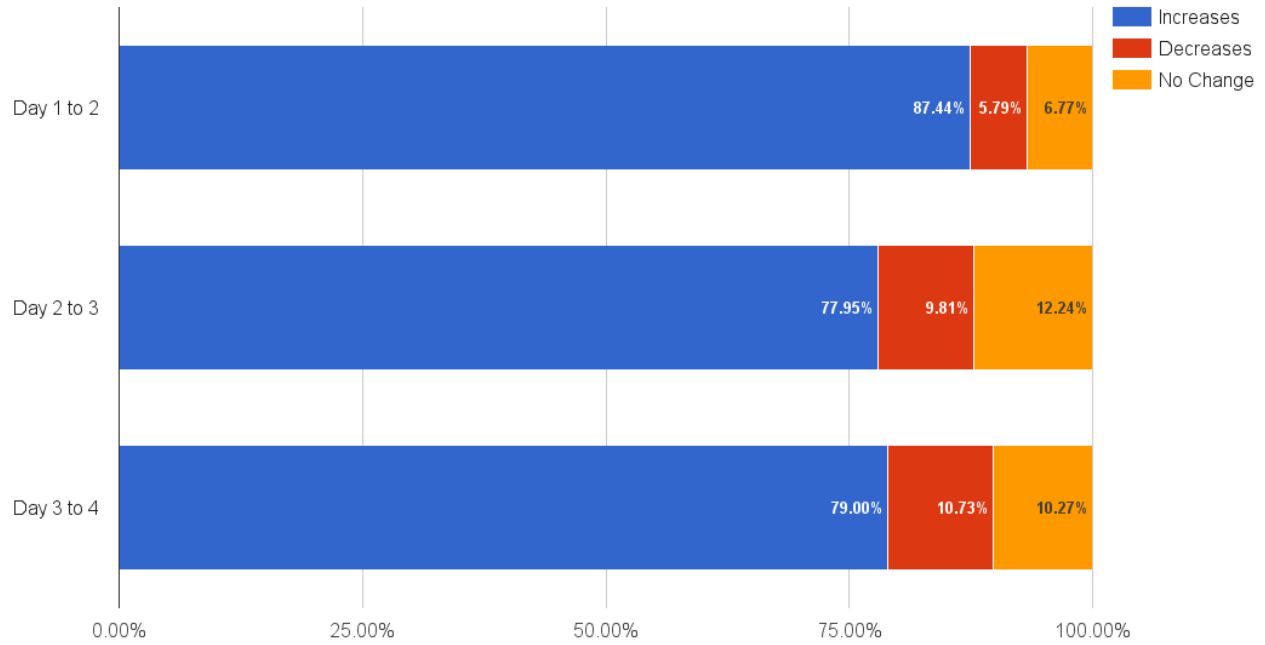
Figure 2.1: Proportion of programs where the cutoff values increased, decreased or did not change from one day to the next

to which a student has no chance of being accepted and, moreover, leads students not to choose programs that they prefer and to which they would actually end up being accepted.

Figure 2.1 shows the proportion of the programs at which the cutoffs increased, decreased or did not change from one day to the next. Some important facts to note are:

- The proportion of programs in which the cutoffs decreased from one day to the next is surprisingly high.

- The proportion of programs in which the cutoffs decreased from one day to the next increased over time.

- More than 10% of the final cutoffs were lower than those communicated to the students on the last day in which they made choices.

In all but five of the 25,686 programs available, the cutoff values by the end of day four were above zero. Figure 2.2 shows the histogram of the values of the cutoffs after they increased or decreased for each day. Although we cannot say that the distributions of cutoffs which decreased and those which increased are not distinguishable, it seems clear that the decreases or increases are not clustered around different values of cutoffs. This indicates that

Decreased: Day 1 to 2   Decreased: Day 2 to 3   Decreased: Day 3 to 4

Increased: Day 1 to 2   Increased: Day 2 to 3   Increased: Day 3 to 4
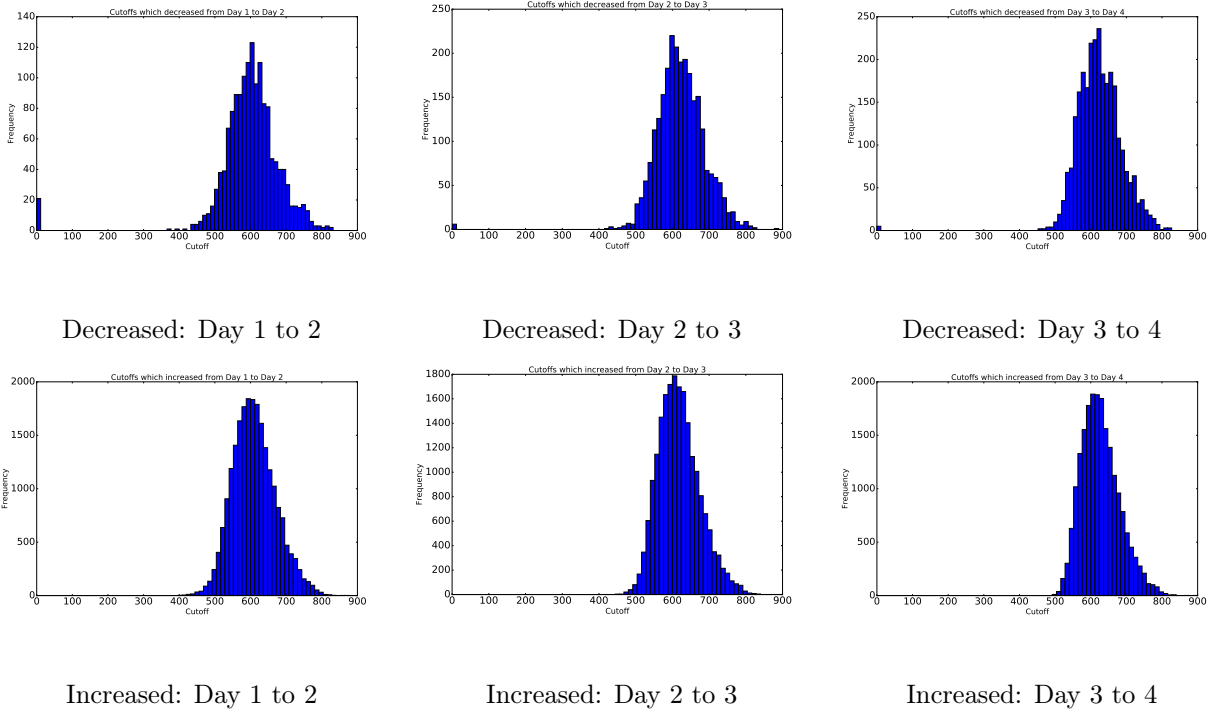
Figure 2.2: Cutoff values after they were decreased/increased from the previous day

the problems identified with the reliability of the cutoffs are not concentrated on more or less competitive programs.

The next question is whether the changes in cutoff grades, when they decrease, are large enough to affect students' beliefs and outcomes. If a cutoff decreases by a very small amount, for example, it may be that no student could have been negatively affected by that change, since the number of those who would be able to choose that program due to that decrease is small or even zero.

The measure that we use to evaluate the degree to which a cutoff value decreases is the change in the value of the empirical cumulative distribution function (CDF), for each program, from one day to the next. For example, say that the cutoff value at program p decreased from day 1 to day 2 from 550 to 500. If the values of the empirical CDF of the cutoffs of program p day are 0.3 and 0.2 for days one and two, respectively, then 30% of all program cutoff values were below the one for program p on day one, but only 20% of them were below the cutoff value of program p on day two.

Figure 2.3 shows the frequency of the changes in the value of the empirical CDF for each pair of consecutive days.[6] For the programs that had their cutoff value reduced between these

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[6]All changes in the value of the CDFs were negative except for one, which had a change below 0.001 and
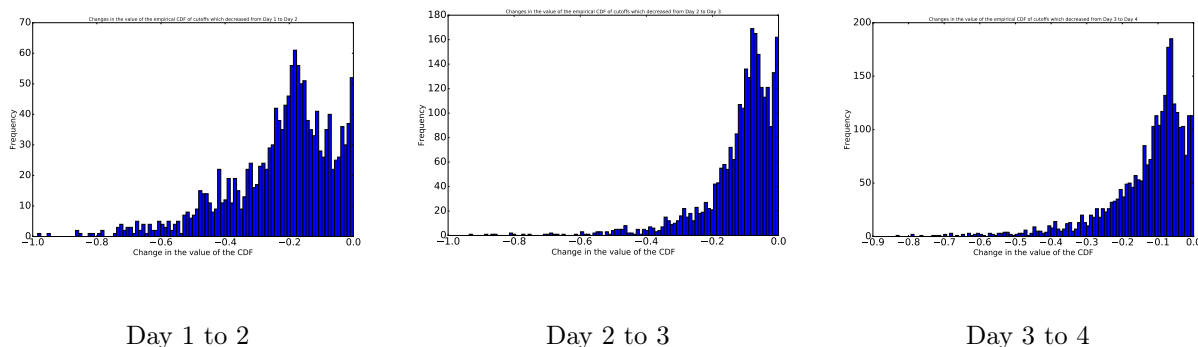
| Day 1 to 2 | Day 2 to 3 | Day 3 to 4 |

Figure 2.3: Change in the value of the empirical CDF for cutoff values that decreased

days, the graphs show that although the largest changes take place from day one to day two, in all cases the proportion of large changes in the ranking is quite substantial. In fact, the percentage of programs where the change in the value of the CDF was lower than -0.2 was 46.87%, 14.61%, and 19.39% for Day1/Day2, Day2/Day3 and Day3/Day4, respectively.

We can therefore conclude that the daily cutoff values which result from candidates interacting with the SISU mechanism fail to provide reliable information about those programs for which a student would not be accepted, since many of them are substantially reduced from one day to the next.

2.2 Manipulations via cutoffs

Other than the fact that under the SISU mechanism the cutoff values do not represent reliable information regarding the chances a student has of being accepted into a college, that mechanism is also subject to what we denote as *manipulation via cutoffs*. A manipulation via cutoffs occurs when a group of students artificially increase the cutoff values of some college, as a way to prevent applications from other students, and then in the last day vacate those seats so that students with a lower exam grade, aware of that manipulation, then take their places. In what follows, we will make the assumption that students have strict preferences over the colleges, and that they follow a simple behavior: a student is said to present *straightforward behavior* if in every step she chooses her most preferred college among those which have cutoff values lower than her grade. The example below shows how manipulations via cutoffs can happen.

Example 1 (Manipulation via cutoffs). Consider the set of students $S = \{s_1, s_2, s_3, s_4\}$ and of colleges $C = \{c_1, c_2, c_3\}$, each with capacity $q_i = 1$ and minimum score zero. Students'

was removed from the graphs for convenience.

preferences are as follows:

$$P_{s_1} : c_1 \ c_2 \ c_3$$
$$P_{s_2} : c_1 \ c_2 \ c_3$$
$$P_{s_3} : c_1 \ c_2 \ c_3$$
$$P_{s_4} : c_2 \ c_1 \ c_3$$

Students' exam grades at the colleges are as follows:

	c_1	c_2	c_3
s_1	100	100	100
s_2	200	200	200
s_3	300	300	300
s_4	400	400	400

Suppose that the SISU mechanism is used, and students present straightforward behavior. The cutoff values, at the end of each day would then be as follows (remember that the cutoffs at $t = 4$ represent the final allocation cutoffs):

	c_1	c_2	c_3
$t = 1$	300	400	0
$t = 2, 3, 4$	300	400	200

The matching produced will therefore be μ:

$$\mu = \begin{pmatrix} c_1 & c_2 & c_3 & \varnothing \\ s_3 & s_4 & s_2 & s_1 \end{pmatrix}$$

Suppose, however, that students s_1 and s_4 modify their behavior, and act instead as follows:

- During $t = 1, 2, 3$, student s_1 chooses college c_3 and student s_4 chooses college c_1.

- In day $t = 4$, student s_1 chooses college c_1 and student s_4 chooses college c_2.

Assuming that the other students present straightforward behavior, the cutoff values at the end of each day would be as follows:

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	c_1	c_2	c_3
$t = 1$	400	$---$	100
$t = 2$	400	300	100
$t = 3$	400	300	200
$t = 4$	100	400	200

The matching produced will be μ':

$$\mu' = \begin{pmatrix} c_1 & c_2 & c_3 & \varnothing \\ s_1 & s_4 & s_2 & s_3 \end{pmatrix}$$

Student s_1 is better off under μ' than under μ, while s_4 is matched to the same college in both cases.

In other words, manipulations via cutoffs consist of a set of students S^H "holding" seats for some time in colleges and "releasing" them so that a set of students S^T can take them in the last day. In order for these manipulations to have a good chance of success, some conditions need to be satisfied.

First of all, the set of students S^H needs to be large enough when compared to the capacity of the college, and their exam grades in that college must also be high compared to other students interested in it. If the number of students in S^H is low when compared to the capacity of the college, the effect of them choosing that college in the value of the cutoff may be much less noticeable. To see that, consider the case in which, at a certain day, there are 100 students choosing college c, which has a capacity of 10 students, and for simplicity assume that those students' scores fill the range $\{1, 2, \ldots, 100\}$ (that is, one student has a score 1, one has a score 2, etc). Then, given those choices, the students who will be tentatively accepted are those with scores 91 to 100, and therefore the cutoff value in that day will be 91. Suppose that S^H has five students, with exam grades $\{300, 301, 302, 303, 304\}$. These are, of course, significantly higher than those of the other students. If all of them choose college c in addition to the 100 students, all of them will be tentatively accepted in that day, but the change in the cutoff value will not be as significant: it will change from 91 to 96. If the capacity of the college was five, the change in the cutoff would instead be from 96 to 300.[7]

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[7]It's not necessarily the case that the number of students in S^H has to be equal to the college's capacity for the change in cutoff to be significant. Consider the case in which the exam scores of the 100 students choosing c are, instead, $\{252, 251, 250, 100, 99, 98, \ldots 4\}$, and the capacity is still five. The cutoff value for college c would be 99 in that day. If S^H has only two students, with exam grades $\{300, 301\}$, them choosing c would lead the cutoff grade at c to change from 99 to 250, instead.

Second, the students who are not in the coalition have to respond in a straightforward way to the cutoff values *in the last day*. This can be considered a reasonably mild requirement. It does not require that the other students follow the straightforward behavior in all days, but only that they do not choose, in the last day, a college where the cutoff value is above their grade in that college.

One may wonder how realistic the first condition is. After all, colleges typically accept hundreds or thousands of students every year, and a coalition of hundreds of high-achieving students performing these potentially risky manipulations does not seem realistic. In many countries (including Brazil and China), however, students apply directly to specific programs in the universities, so even though the universities as a whole accept hundreds or thousands of students, the number of seats at each program is often below 100, and many times lower than 30 or 20. Moreover, even those seats are often subdivided. In China, the seats in each program are partitioned between seats reserved for candidates from specific provinces. In Brazil, federal universities partition the seats in the programs into five sets of seats, reserved for different combinations of ethnic and income characteristics. Finally, universities sometimes offer only a subset of the total number of seats in a program through the centralized matching process. In fact, the median number of seats offered in each option available during the January 2016 selection process in Brazil, where more than 228,000 seats in public universities were offered, was *five*.

Moreover, there is evidence that this type of manipulation takes place in real life. In the Chinese province of Inner Mongolia, a mechanism which has some similarities to the SISU mechanism is used to match students to programs in universities.[8] While the mechanism itself has significant differences, it is also vulnerable to manipulation via cutoffs. This fact seems to be exploited by students, as documented by China News:[9]

> "(...) in fact, since 2008, the clearinghouse found that some high scored students applied to a college with lower cutoff score. For example, their score allows them to go to PKU or Tshinghua, but they chose Beijing Polytech first. On the other hand, some other students, from the same high school often, applied to college that their score would not allow them to go initially (...) [the] system shows that their rank is below the capacity — so they can't be admitted under usual terms — however they do not revise their choices."

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[8]For detailed descriptions of the mechanism, see Gong and Liang [2016] and Chen and Pereyra [2016].

[9]Source (in Chinese): http://www.chinanews.com/edu/2014/09-04/6562740.shtml (Accessed on 09/12/2017)

Even more remarkably, there seems to be evidence that high schools are coordinating students' actions:

> "(...) the clearing house noticed that, 2 or 3 min before the deadline, the ranking of students in the system is changing – this is the evidence that high schools are organizing their own high scored students to occupy seats for low scored students."

3 The Generalized Iterative Deferred Acceptance Mechanism

In this section, we introduce the generalized version of our proposed mechanism, inspired by the SISU, denoted the Generalized Iterative Deferred Acceptance Mechanism (GIDAM). In this generalized version, we consider a more general setup, in which the criteria used by colleges may be more general than one simply based on exam grades, allowing, for example, for the use of affirmative action policies or variations in financial aid in student admissions.[10] This version also allows for the same students and colleges to be matched under different contractual terms, as in the matching with contracts model introduced by Hatfield and Milgrom [2005]. Finally, we also allow for students not having to only submit one choice at a time, but also rankings over the available options.

The underlying principle behind the SISU mechanism, however, is maintained: instead of asking students to submit a preference ranking over all the available options, students are sequentially asked for partial information about the outcomes that they would like to obtain, and receive information on the feasibility of those options over time. A special case of GIDAM, which considers the same setup of college admissions presented while introducing the SISU mechanism, is introduced in subsection 3.1.

A **matching with contracts market** is a tuple $\langle S, C, T, X, P_S, F_C \rangle$:

1. A finite set of **students** $S = \{s_1, \ldots, s_n\}$,

2. A finite set of **colleges** $C = \{c_1, \ldots, c_m\}$,

3. A vector of **contractual terms** $T = (t_1, \ldots, t_\ell)$,

4. A set of **valid contracts** $X \subseteq C \times S \times T \cup \{\varnothing\}$,

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[10]See, for example, Hafalir et al. [2013], Aygün and Bó [2013], Shorrer and Sóvágó [2017], Hassidim et al. [2015], Yenmez.

5. A list of strict **student preferences** $P_S = (P_{s_1}, \ldots, P_{s_n})$ over $X \cup \{\varnothing\}$[11], and the respectively derived weak preferences R_S,

6. A list of **college choice functions** over sets of students $F_C = (f_{c_1}, \ldots, f_{c_m})$, where for every $c \in C$ and $I \subseteq X$, $f_c : 2^X \to 2^X$, $\{(c, s, t), (c, s', t')\} \subseteq f_c(I) \implies s \neq s'$ and $(c', s, t) \in f_c(I) \implies c' = c$.

For any $I \subset X$, $s \in S$ and $c \in C$, denote $I_s \equiv \{(c, s', t) \in I : s' = s\}$, $I_c \equiv \{(c', s, t) \in I : c' = c\}$, $s(I) \equiv \{s \in S : \exists (c, s, t) \in I\}$ and $c(I)$ be defined analogously. We abuse notation and let $c(x)$ and $s(x)$ be the college and student in contract x, respectively. An **outcome** is a set of contracts $Y \subseteq X$ such that Y contains at most one contract per student, that is, $|Y_s| \leqslant 1$ for each $s \in S$. Denote by $\mathcal{X}$ the set of all outcomes. An outcome Y is **individually rational** if for every student s, $Y_s R_s \varnothing$ and for every college c, $Y_c = f_c(Y_c)$. Define by **maximum rank function** a function $\pi : \mathbb{Z}^+ \to \mathbb{N} \cup \{\infty\}$ which defines, for each step $t = 0, \ldots, T^{Max}$, what is the maximal length of a ranking that a student may submit.

The **GIDAM mechanism** consists of the following steps:

$t = 0$ A weakly informative signal about the set of feasible allocations is broadcast. Additionally, each student is given an individualized menu of contracts, consisting of the contracts involving said student that colleges deem acceptable.[12] Each student who is given a non-empty menu is asked to submit an ordered list with at most $\pi(0)$ contracts in their menu. After all students submit their lists (or opt not to,) these are used to perform a cumulative offer process. That is, students one at a time offer their highest ranked contract, in the list submitted, to the college involved in it, and colleges choose among all contracts offered, cumulatively, with their choice functions. Whenever a contract is rejected, the student involved in it offers the next highest-ranked contract, if any. The step ends whenever every student has a contract being held by a college or had all of those in the list submitted rejected.

$0 < t \leqslant T^{Max}$ At the beginning of the step, a weakly informative signal about the set of allocations that are still feasible is broadcast. Additionally, each student who doesn't have a contract being held by a college is given an individualized menu of contracts, consisting of the contracts involving said student that colleges would accept, while

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[11]In some places we abuse notation and use P_s also over sets with only one contract. Here, $\varnothing$ represents the null contract, representing a student remaining unmatched to any college. We also assume that a student's preference is over contracts in which she is involved and $\varnothing$.

[12]That is, these contracts would be chosen if each one of them were the only option given to the college involved in it.

having all contracts that were offered in previous steps still available. Each student who is given a non-empty menu is asked to submit an ordered list with at most $\pi(t)$ contracts in their menu. The same cumulative offer process is undertaken, as in the previous step, where the lists submitted by the students in this and previous steps are used to offer contracts to colleges, which are considered together with those offered in previous steps.

The process ends after the step $t = T^{Max}$ or whenever the set of contracts being held by all colleges doesn't change from one step to the next. Denote that last step by T^*. A formal definition of the mechanism can be found in the Appendix.

The public signals do not play a role in the results we present, but in general they may affect other incentives induced by the mechanism and be useful in terms of transparency. In subsection 3.1, for example, the public signals will be the cutoff grades at each college, as in the SISU mechanism.

Example 2. Consider a matching with contracts problem in which there are four colleges $C = \{c_1, c_2, c_3, c_4\}$, each with one seat available, and four students $S = \{s_1, s_2, s_3, s_4\}$. Colleges may accept students with or without financial aid. Colleges always prefer to accept students without financial aid, and select them based on their grades in a single national exam otherwise. Students' grades in the national exam follow their indexes: s_1 has the highest grade, s_2 the second-higest, etc. Let the maximum rank function be such that $\pi(t) = 2$ for every $t \geqslant 0$ and $T^{Max} = \infty$.

The table below shows, for each student, the menu of contracts offered in the first step (which, for all students, contain all possible contracts with colleges), and a list that is submitted by each student in the first step. We represent contracts with financial aid with the letter F, and without financial aid with N.

Student	Tentative match	Menu offered	List submitted
s_1	$\varnothing$	C_1 [F|N] C_2 [F|N] C_3 [F|N] C_4 [F|N]	C_1 [F|] $>_2$ C_4 [F|]
s_2	$\varnothing$	C_1 [F|N] C_2 [F|N] C_3 [F|N] C_4 [F|N]	C_1 [F|] $>_2$ C_2 [F|]
s_3	$\varnothing$	C_1 [F|N] C_2 [F|N] C_3 [F|N] C_4 [F|N]	C_2 [F|] $>_3$ C_2 [|N]
s_4	$\varnothing$	C_1 [F|N] C_2 [F|N] C_3 [F|N] C_4 [F|N]	C_1 [F|] $>_4$ C_2 [F|]

The table below shows the menus offered in the next step, and choices that we consider students make. Notice that the number of options in the menus offered to students s_2 and s_4 is different. Since s_4 has a low exam grade, no contract with C_2 would be accepted anymore. Also, while the student with the highest exam grade is tentatively matched to college c_1, a contract without financial aid is offered in the menu to student s_2.

Student	Tentative match	Menu offered	List submitted
s_1	C_1 [F|]	$\varnothing$	$\varnothing$
s_2	$\varnothing$	C_1 [|N] C_2 [|N] C_3 [F|N] C_4 [F|N]	C_1 [|N] $>_2$ C_4 [F|]
s_3	C_2 [|N]	$\varnothing$	$\varnothing$
s_4	$\varnothing$	C_1 [|N] C_3 [F|N] C_4 [F|N]	C_1 [|N] $>_4$ C_4 [F|]

In step $t = 3$, only student s_4 will be given a menu, with only three contracts. This happens even though s_1 had her previous match to C_1 rejected. The mechanism continued down her list submitted in step $t = 0$ and matched her to C_4 with financial aid:

<center>Step $t = 3$</center>

Student	Tentative match	Menu offered	List submitted
s_1	$\begin{array}{c}\boxed{C_4}\\ \boxed{F}\end{array}$	$\varnothing$	$\varnothing$
s_2	$\begin{array}{c}\boxed{C_1}\\ \boxed{N}\end{array}$	$\varnothing$	$\varnothing$
s_3	$\begin{array}{c}\boxed{C_2}\\ \boxed{N}\end{array}$	$\varnothing$	$\varnothing$
s_4	$\varnothing$	$\begin{array}{c}\boxed{C_3}\\ \boxed{F\,N}\end{array}\ \begin{array}{c}\boxed{C_4}\\ \boxed{N}\end{array}$	$\begin{array}{c}\boxed{C_3}\\ \boxed{F}\end{array} >_4 \begin{array}{c}\boxed{C_4}\\ \boxed{N}\end{array}$

Given the submitted list, the final matching is produced by the end of that step:

<center>Step $t = 4$</center>

Student	Final match
s_1	$\begin{array}{c}\boxed{C_4}\\ \boxed{F}\end{array}$
s_2	$\begin{array}{c}\boxed{C_1}\\ \boxed{N}\end{array}$
s_3	$\begin{array}{c}\boxed{C_2}\\ \boxed{N}\end{array}$
s_4	$\begin{array}{c}\boxed{C_3}\\ \boxed{F}\end{array}$

The example above highlights some of the main characteristics of the GIDAM mechanism. Students only have to submit a list when the absence of that information would not allow the mechanism to determine where their next tentative allocation (if any) should be. Contracts that are no longer feasible for a student are not offered in their menus, reducing the number of options that she has to consider.

We can define a **matching function** that represents the outcome, μ, can be defined such that, for every $c \in C$, $\mu(c) = f_c\left(A^{T^*}(c)\right)$, and for every $x \in \mu(c)$, let $\mu(s(x)) = x$. Whenever $T^{Max} = \infty$ or there is a $t \leqslant T^{Max}$ such that $\pi(t) = \infty$, we say that the GIDAM

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is **unbounded**. When GIDAM is unbounded, therefore, a student is able to express, either over time or via a ranking in some steps, a sequence of choices over as many contracts as she wishes.

A **random outcome** is a probability distribution over $\mathcal{X}$. An outcome $X' \subseteq X$ is **stable** if it is individually rational and there is no college c and set of contracts $X'' \subset X$ such that $X'' \neq f_c(X')$, $X'' = f_c(X' \cup X'')$ and for every $s \in s(X'')$, $xR_sX'_s$. More specifically, we say that a student s and college c form a **blocking pair** under X' if s has a contract in $X'' \backslash X'$. A stable outcome is called the **student-optimal stable allocation** if every student weakly prefers it to any other stable outcome.

In order to guarantee that the outcomes and incentives of the GIDAM mechanism satisfy desirable properties, it is necessary to impose some restrictions on the colleges' choice functions. The first one comes from Hatfield and Kojima [2010]:

Definition 1. Contracts in X are **unilateral substitutes** for college c under f_c if there do not exist contracts $x, z \in X$ and a set of contracts $Y \subseteq X$ such that $s(z) \notin s(Y)$, $z \notin f_c(Y \cup \{z\})$ and $z \in f_c(Y \cup \{x, z\})$.

Another condition that we use comes from Aygün and Sönmez [2013]:

Definition 2. The choice function f satisfies **irrelevance of rejected contracts (IRC)** if $x \notin f(X' \cup \{x\})$ implies $f(X' \cup \{x\}) = f(X')$ for all $X' \subset X$ and $x \in X \backslash X'$.

Finally, the last property that will be used was introduced in Hatfield and Milgrom [2005]:

Definition 3. The choice function f satisfies the **law of aggregate demand** if for all $Y \subseteq Z$, $|f(Y)| \leqslant |f(Z)|$.

Lemma 1. *Assume that for every college $c \in C$, f_c satisfies IRC and contracts in X are unilateral substitutes. Then, for every student s and $0 \leqslant t \leqslant t' \leqslant T^*$, if the set of contracts in the menu given to s in step t is non-empty, then all contracts in a menu given in step t' are also in the one given in step t.*

What Lemma 1 says is that once a contract becomes unavailable for a given student, that contract will never become available again, regardless of the strategies used by the students. This shows that one information given by the mechanism after each step – the set of acceptable contracts available to the student – constitutes reliable information about the contracts which are not available anymore for a student, as opposed to the SISU mechanism. We define a formal generalization of the "straightforward behavior" [Roth and Sotomayor, 1992] when interacting with the GIDAM mechanism:

Definition 4. A strategy of student $s \in S$ is **straightforward with respect to** P^* if for every step t in which a non-empty menu $\psi^t(s)$ is offered by the mechanism, s submits a ranking with the top $k(\psi^t(s), t)$ contracts in $\psi^t(s)$, ordered as in P^*, where $k : 2^X \times \mathbb{N} \to \mathbb{N}$ is a function such that, for every t, $1 \leqslant k(\cdot, t) \leqslant \pi(t)$, and $k(\cdot, t^\infty) = |\psi^{t^\infty}(s)|$, where t^∞ is the highest value of t such that $\pi(t) = \infty$.

A strategy is straightforward in the GIDAM, therefore, when in every step the student submits either her full preference over the contracts in the menu offered (whenever $\pi(t)$ is large enough) or some *truncation* of her true preference. Moreover, when there are multiple steps in which a student can submit an unbounded ranking over contracts, she should rank all those offered at least in the last step in which that is allowed. When $\pi(t) = 1$ for all t, the definition reduces to the definition of straightforward behavior in Roth and Sotomayor [1992]: everytime the student is asked, she picks the most preferred alternative. When students follow straightforward strategies, the outcome produced by the unbounded GIDAM is of a well-known type:

Proposition 1. *Assume that, for every college $c \in C$, f_c satisfies IRC and contracts are unilateral substitutes. If all students' strategies are straightforward with respect to P_S, there is a finite number of steps T^* after which the outcome of any unbounded GIDAM mechanism is the student-optimal stable outcome with respect to P_S.*

The proof of Proposition 1 is based on the fact, shown in Hirata and Kasuya [2014], that the cumulative offer process that takes place during the GIDAM mechanism is *order independent* and when students follow straightforward strategies the outcome is the student-optimal stable outcome. As a result, all combinations of such strategies yield the same result.

When colleges' choice functions satisfy IRC, unilateral substitutes and the law of aggregate demand, a direct mechanism that produces the student-optimal stable outcome is strategy-proof [Aygün and Sönmez, 2013, Hatfield and Kojima, 2010]. That is, submitting her true preference ranking over contracts is a weakly dominant strategy for every student. One may be tempted to conclude that this will imply that straightforward strategies, which are the equivalent of truth-telling in this dynamic setting, are also dominant under the GIDAM mechanism, but the proposition below shows that not only is this not the case, but that students may not have any dominant strategy at all.

Proposition 2. *A student may not have a weakly dominant strategy under the GIDAM mechanism.*

The reason why not following a straightforward strategy may be profitable is that, in contrast to the case of the direct mechanism, an agent may influence others' actions by modifying the signals received by them. So if, for example, a student has a strategy that depends in some way on the signals produced by one's actions, or even on the sequence of menus that are presented or timing of the rejection in a particular choice, that fact could be exploited. We will show, however, that the strategy profile in which students follow straightforward strategies constitutes a robust equilibrium. The equilibrium concept that we use is a refinement of the Perfect Bayesian Equilibrium.

Definition 5. A strategy profile is an **ordinal perfect Bayesian equilibrium (OPBE)** if, at every information set, every deviation from the equilibrium strategy is stochastically dominated by following it.

The definition above is intentionally informal, but its formal version can be found in the appendix. When a strategy profile is an OPBE, therefore, the probabilility of obtaining the most preferred contract, the two most preferred contracts, the three most preferred contracts, etc., is weakly greater when following the equilibrium strategy when comparing to any deviating strategy, when starting from any step. Equivalently, no deviating strategy yields a better expected utility, for any utility function that represents the students' ordinal preferences. For that, we consider the extensive-form game induced on the students by the GIDAM mechanism. We will allow students to have uncertainty about other students' preferences and exam grades. The sequence of events is as follows:

1. Step $t = 0$: Nature draws the values of X and P from a joint distribution f, and each student s observes the realization of X_s and P_s.

2. Steps $1 \leqslant t \leqslant T^{Max}$: students interact with the GIDAM mechanism. That is, in each step t, every student $s \in S$ receives a menu of contracts and a maximum rank value $\pi(t)$ and has to submit a ranking over those contracts, as in the description of the mechanism above. At the end of the step, the public signals are observed by all students. The mechanism terminates at some step $T^* \leqslant T^{Max}$.

3. Step $T + 1$: Students are matched to their outcomes produced by the GIDAM mechanism.

Our main result shows that, when facing that game, students following straightforward strategies is an OPBE.

Theorem 1. *Consider a maximum rank function π and an unbounded GIDAM mechanism under it, and let σ^* be the strategy profile in which all strategies are straightforward with respect to P_S. Then σ^* is an OPBE of the game induced by the GIDAM mechanism.*

The proof of Theorem 1 is fundamentally based on the fact that, although the space of deviating strategies is significantly large, they are all indistinguishable, from the perspective of an observer, from a student following a straightforward strategy for some different preference. This allows us to evaluate deviating strategies in all of their paths, which may include multiple interactions that the student may have with the mechanism. Without this, it is difficult to determine the final outcome of a generally specified deviation.[13] This, however, is not present in other iterative matching mechanisms, such as the SISU[14] and the mechanism used in Inner Mongolia.

3.1 Iterative Deferred Acceptance

Here we introduce the Iterative Deferred Acceptance Mechanism (IDAM), which is an application of the GIDAM mechanism for the problem of college admissions tackled by the SISU mechanism, in which the criterion used by colleges to select candidates is based solely on their grades in an exam, and therefore acceptability may be summarized by cutoff grades. A **exam-based college matching market** is a tuple $\langle S, C, q, P_S, z, \underline{Z} \rangle$:

1. A finite set of **students** $S = \{s_1, \ldots, s_n\}$,

2. A finite set of **colleges** $C = \{c_1, \ldots, c_m\}$,

3. A **capacity vector** $q = (q_{c_1}, \ldots, q_{c_m})$,

4. A list of strict **student preferences** $P_S = (P_{s_1}, \ldots, P_{s_n})$ over $C \cup \{s\}$[15],

5. A list of **vectors of exam scores** $z = (z(s_1), \ldots, z(s_n))$, where for each $s \in S$, $z(s) = (z_{c_1}(s), \ldots, z_{c_m}(s))$, are the exam scores that student s obtained, respectively, at college $c_1 \ldots, c_m$. We assume that for every $s, s' \in S$ and $c \in C$, $z_c(s) = z_c(s') \implies s = s'$,

6. A list of **minimum necessary scores** $\underline{Z} = (\underline{z}_{c_1}, \ldots, \underline{z}_{c_m})$.

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[13]In fact, in most sequential matching mechanisms in the literature (for example, Alcalde and Romero-Medina [2005], Triossi [2009], and Romero-Medina and Triossi [2014]) the number of times an agent interacts with the mechanism is either exogenously given or is one in equilibrium.

[14] See Remark 4 in the appendix.

[15]Here s represents a student remaining unmatched to any college.

The set of contracts, colleges' choice functions and public signal functions are derived from the above as follows:

- The set of valid contracts is $X = \{(s, c, z_c(s)) : s \in S, c \in C \text{ and } z_c(s) \geqslant \underline{z}_c\}$. That is, the valid contracts are between all colleges and the students who have an exam grade at least as high as the minimum at that college. The contractual term is the exam grade itself. Since there is only one contract between each student and college, we can refer to the menus of contracts as simply being menus of colleges. Moreover, students' preferences over contracts are directly derived from the preferences over colleges.

- For a given set of contracts Y, colleges' choice functions f_c select the top q_c contracts in Y_c with respect to z_c if $|Y_c| \geqslant q_c$, and all contracts in Y_c otherwise.

- The public signal function ζ yields colleges' cutoffs (the lowest exam grade in that college necessary to be chosen, given the current set of contracts being held). Specifically, $\zeta(Y) = (\zeta_{c_1}(Y), \ldots, \zeta_{c_m}(Y))$, where $\zeta_c(Y) = \underline{z}_c$ when $|Y_c| < q_c$ and $\zeta_c(Y)$ is the q_c'th highest value of z_c in the contracts in Y_c.

It is easy to see that f_c, as defined above, satisfies unilateral substitutes and the law of aggregate demand. This implies that Proposition 1 and Theorem 1 also hold for the IDAM mechanism.

Corollary 1. *If students follow straightforward strategies, the outcome of the IDAM mechanism is the student-optimal stable outcome.*

Corollary 2. *All students following straightforward strategies with respect to their true preferences is an OPBE of the game induced by the IDAM mechanism.*

Let ζ_c^t be the value of the cutoff of college c made public in step t, as defined above. In light of the definition of f_c, Lemma 1 leads to the following conclusion:

Corollary 3. *(Cutoff grades never go down) For every $0 \leqslant t \leqslant T^*$ and $c \in C$, $\zeta_c^t \geqslant \zeta_c^{t-1}$.*

As described in section 2.2, manipulations via cutoffs consist of temporarily inflating the cutoff value of a college and then reducing it. The corollary above implies, therefore, that manipulation via cutoffs is not feasible:

Remark 3. The IDAM mechanism is not manipulable via cutoffs.

It is also worth noting that, while the process that takes place during the execution of the IDAM mechanism resembles a "worker-proposing" version of the salary adjustment in Kelso and Crawford [1982], our results shows that, at least when students are the only strategic agents, the *process itself* is an equilibrium when interactions are restricted in the way defined by the IDAM mechanism.

4   Time feasibility

The number of steps that it takes for the GIDAM or IDAM mechanisms until it produces the Student Optimal Stable Outcome when students follow the straightforward strategy depends on the interaction of multiple variables, such as students' preferences, the maximum rank function, the colleges' choice functions (or students' exam grades), etc. It is very important, however, to have some sense of how many steps that will take, what happens if the value of T^* is not large enough, and if there are other viable alternatives to reduce the time it takes to produce the outcome.

For the results below, we consider exam-based college matching markets where the set of students and colleges can be partitioned as $S = \left\{ S^1 \cup S^2 \cup \cdots \cup S^k \right\}$ and $C = \left\{ C^1 \cup C^2 \cup \cdots \cup C^k \right\}$, where $\sum_{c \in C^i} q_c \leqslant |S^i|$ and students in S^i have higher grades at colleges in C^i than those not in S^i. This is consistent with situations in which college exams have math and literature sections and students are good at either math or literature. Notice, however, that when $k = 1$ this definition accommodates any market in which the number of seats in colleges does not exceed the number of students.

One related question that we tackle in the proposition below is how "far" from a stable matching the outcome will be if we use the bounded IDAM mechanism for a number of steps smaller than that necessary to produce the Student Optimal Stable Matching but with students still following the straightforward strategy. The measure of distance from a stable matching that we use is the number of individuals involved in blocking pairs.

Proposition 3. *Let, for every $i \in \{1, \ldots, k\}$, $c, c' \in C^i$ and $s, s' \in S^i$, $z_c(s) > z_c(s') \iff z_{c'}(s) > z_{c'}(s')$ and for all $s \in S^i$, $c \in C^i$ and $c' \notin C^i$, cP_sc'. If students follow straightforward strategies then:*

1. *The maximum number of steps it takes for the unbounded IDAM mechanism to produce the student optimal stable outcome is $\max_i \{|C^i|\}$.*

2. *If the IDAM mechanism runs for $T^* < \max_i \{|C^i|\}$ steps, the maximum number of*

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individuals involved in blocking pairs is $n - \sum_{j=1}^{k} \sum_{i=1}^{T^*} q_i^j$, where for each j, $q_1^j \leqslant q_2^j \leqslant \cdots \leqslant q_{|C^j|}^j$ is the ordering of the capacities of the colleges in C^j.

The configuration of preferences used in Proposition 3 is consistent with scenarios in which the top preferences are mutually partitioned between students and colleges, and colleges share the selection criteria among their top students. One example would be a college admissions program that is based on national exams consisting of questions on different subjects and college programs that rank the students based on their grades in those different subjects. The stronger assumption in this case is that the partition is such that students are among the best at only one of the subjects. For example, if the partitioning of college programs is between medical sciences, STEM, and humanities, a student who is among the top at humanities is not so at STEM or medical subjects.

For the case of common grades between all colleges, the result does not have to rely on any assumption on students' strategies.

Proposition 4. *If grades are common across colleges, the maximum number of steps it takes for the unbounded IDAM mechanism to produce the student optimal stable outcome is m.*

4.1 The IDAM+DA alternative

One possibility that a policymaker could adopt is to use a hybrid between the iterative mechanisms considered here and the traditional deferred acceptance, which we denote by IDAM+DA. It consists of running the IDAM mechanism, with students making only one choice at a time, for a fixed number of steps, and then asking students to submit a ranking over the remaining options available. Formally, for a given number of steps $k > 0$, the IDAM+DA is simply defined as the IDAM mechanism in which the maximum rank function is such that for all $t \in \{1, \ldots, k\}$, $\pi(t) = 1$, and $\pi(t+1) = \infty$.

One of the main advantages of the IDAM+DA is that it ends after a number of steps set by the designer: $k+1$. Moreover, being an unbounded IDAM, it implements the student-optimal stable outcome in an OPBE of straightforward strategies. The additional advantage is that, as we will show in the following subsections, these initial iterative steps done before the "DA-like" last step will clear a large part of the market, leaving a smaller residual allocation problem to be solved. More specifically, after the initial steps, the number of students who still need to have their match determined, and the number of options that remain available for those whose matches still need to be determined, are both substantially reduced.

The IDAM+DA mechanism has some similarities to the college admissions procedure in Germany [Grenet et al., 2017]. In that system, there is an initial stage, which lasts a specific number of days, in which colleges send offers to students who listed them as acceptable. Students may accept or reject these offers, which may lead to additional offers being sent. After that step ends, the remaining seats and students are allocated using the college-proposing deferred acceptance mechanism [Gale and Shapley, 1962] using rankings with limited length. While this has significant differences from the one we propose here, it shows that policymakers are already experimenting with procedures that mix a sequential phase which clears part of the market with a final one that is supposed to clear the rest of it.

4.2   A model with a continuum of students

In this section we introduce a model with a continuum of students and a finite number of colleges, which allows us to obtain some approximations that can be helpful on the issue of time feasibility. This model is based on the logit model with preferences in Ashlagi and Shi [2014]. There is a unit mass of students. Colleges share a common grading of students: each student's grade is drawn from the uniform distribution with support $[0, 1]$, and have a minimum grade of zero. Each college has capacity $q > 0$, and the total mass of seats is weakly smaller than the number of students, that is, $q \times m \leqslant 1$. Students' ordinal preferences over colleges are derived from a random utility model, in which a student s's utility from being matched to a college c is:

$$u_{sc} = \alpha \nu_c + \varepsilon_{sc}$$

Students' utilities thus consist of a common value, shared by all students $(\alpha \nu_c)$, where $\alpha > 0$, and an idiosyncratic shock drawn from a standard Gumbel distribution. The value of α, therefore, functions as a parameter for the correlation of preferences: the larger it is, the more students' utilities depend on the common value as opposed to their idiosyncratic shocks. For simplicity, we use a linear value function $v_{c_i} = (m - i)$. Therefore $\nu_{c_1} > \nu_{c_2} > \cdots > \nu_{c_m}$. Denote $r_i = e^{\alpha \nu_{c_i}}$. Due to the logit utility structure, for every student, the probability that college c_i is the most preferred is:

$$P^1(c_i) = \frac{r_i}{\sum_{j=1}^m r_j}$$

Clearly, therefore, $P^1(c_2) > P^1(c_3) > \cdots > P^1(c_m)$. Consider now the IDAM+DA mechanism. If students follow straightforward strategies, the number of students who apply

to college c_i is $P^1(c_i)$ in step $t = 1$. If there is a college c_j such that $P^1(c_j) < q$, therefore, the number of students who choose c_j (or any c_{j+k}) is smaller than the capacity, and therefore at the end of the first step all of those colleges will have zero as their cutoff values. Suppose here, for ease of exposition, that $P^1(c_m) \geqslant q$, so that no cutoff will be zero. The cutoff value $\zeta^1_{c_i}$ will be, for those which are above zero, the lowest grade from a mass of q top students among $P^i(c_i)$ students:

$$\zeta^1_{c_i} = 1 - \frac{q}{P^1(c_i)}$$

It follows, therefore, that the cutoff values, after step $t = 1$ are such that:

$$\zeta^1_{c_1} > \zeta^1_{c_2} \geqslant \cdots \geqslant \zeta^1_{c_m}$$

Notice that, since $\zeta^1_{c_1}$ is larger than all other cutoffs, this implies that all students who had their choices in $t = 1$ rejected have grades below $\zeta^1_{c_1}$ and therefore will not have c_1 in their menus in the next step. Most importantly, Lemma 1 implies that c_1 will not ever appear in their menus again.

Consider the cutoff values that are published after the last day of the execution of the mechanism for each college c, and denote it ζ^*_c. One consequence of Corollary 3 is that after step t, no student with a grade above $\zeta^*_{c_t}$ will be rejected from the last choice she made. This follows from the fact that after a student is rejected, the cutoff value at that college must then become higher than that student's grade. We can say, therefore, that after step t all students with a grade above $\zeta^*_{c_t}$ are *permanently matched*, and will not be presented with any other menu during the iterative part of the IDAM+DA mechanism.

Proposition 5. *Let M^t be the mass of students who are permanently matched to a college by the end of step t. Then:*

$$(i) \ M^1 > \mathbb{M}^1 = \frac{e^{-\alpha(m-1)}(e^{m\alpha} - 1)}{e^\alpha - 1}q$$

$$(ii) \ \textit{For every } 1 \leqslant t < m - 1, \ \Delta M^t > \Delta\mathbb{M}^t = q\left(1 - e^{-\alpha(m-t)}\right)$$

Where $\Delta M^t = M^{t+1} - M^t$. Proposition 5 shows us that each initial step in the IDAM+DA mechanism increases the number of students who will not have to interact with the mechanism again because they are matched to their final match or because they will not have any chance of getting a seat. Moreover, the marginal effect of those increases is larger in the first steps, since $\Delta\mathbb{M}^t > \Delta\mathbb{M}^{t+1}$. The following corollaries can be drawn from the proposition:

Corollary 4. *Regarding the values of $\mathbb{M}^1$ and $\Delta\mathbb{M}^t$:*

$$\frac{\partial \mathbb{M}^1}{\partial \alpha} < 0 \ \ and \ \frac{\partial \left(\Delta\mathbb{M}^t\right)}{\partial \alpha} > 0$$

That is, when preferences are less correlated (and α is therefore smaller), the number of students permanently matched in the first step is higher, whereas later steps may have their rate of permanent matches reduced. This indicates that a great part of the advantage of the IDAM+DA is captured in a few initial steps, and that this benefit is greater the less correlated preferences are. The reason for this is that the less correlated preferences are, the lower the final cutoff value of the most preferred colleges will be. Therefore, the number of students with a grade above that is larger. At the same time, the difference between the cutoffs of the different colleges is smaller, leading to a smaller marginal effect of the additional steps. In the next section we show, via simulations, that this insight also applies in more general settings.

4.3   Simulations

In this section we present the outcome of two sets of simulations. In the first set we compare the number of steps it takes for the unbounded IDAM mechanism with one choice per step to produce the student-optimal stable matching with the length of the rankings that students need to submit so that, for the market in question, truth-telling is an equilibrium. This allows us to have both a quantitative idea of how these two variables relate (time in IDAM vs length of ranking in DA) and in which scenario each mechanism performs relatively better. As we will show, the answer for these questions is surprisingly irregular.

In the second set of simulations, we evaluate the IDAM+DA mechanism, which consists of first running a fixed number of steps of the IDAM with one choice per step and then asking students to submit a full ranking over the options remaining for each student. We show that those initial steps of the IDAM+DA mechanism have the effect of substantially reducing the length of the ranking that is necessary for students to submit at the last step, making it an attractive alternative for real-life applications. These results, therefore, generalize the ones obtained with the model with a continuum of students in section 4.2.

The construction of the problems to be simulated follows a method similar to that applied in Hafalir et al. [2013]. Students' ordinal preferences are derived from utilities that each student has over the colleges. No college is deemed unacceptable by any student. Student $s \in S$'s utility from being assigned to college $c \in C$ is the following:

$$u_s\left(c\right) \;=\; \alpha\Theta^c + \left(1 - \alpha\right)\Theta_s^c$$

The interpretation of the parameters goes as follows. The utility that a student s derives from being assigned to a college c is a combination of a value that is shared by all students (Θ^c) and an idiosyncratic value that is unique to a student-college pair (Θ_s^c). The value of Θ^c could therefore be the widespread understanding of the quality of the college and Θ_s^c incorporate, for example, how the college's characteristics fit the student's particular objectives. For each problem, and for each value of $c \in C$ and $(c, s) \in C \times S$, Θ^c and Θ_s^c are independently drawn from the normal distribution with mean zero and variance 1. The value of α, which represents the correlation of preferences between students, is exogenously set in the range $[0, 1]$.

Students' exam grades at each college follow a similar model, and the grade that student s has at college c is:

$$z_c\left(s\right) \;=\; \beta\Theta^s + \left(1 - \beta\right)\Theta_c^s$$

Here once again, for each problem, the value of Θ^s and Θ_c^s is independently drawn from the normal distribution with mean zero and variance 1. The minimum grade at all colleges is zero (that is, all students are acceptable to all colleges). Moreover, $\beta \in [0, 1]$ is an exogenous parameter which represents the degree of correlation between a student's grades at colleges. Notice that when $\beta = 1$, students have the same exam grade in all colleges. This is the case, for example, when the criterion used for ranking students is the grade in a single national exam.

In each simulation performed, we set the values of the parameters (n, m, q, α, β) (where q is the common capacity for all colleges) and generated 20 problems, each representing independent draws for values of the random variables. Every combination of the values of the parameters α and β, in steps of 0.1, were used. In other words, every $(\alpha, \beta) \in [0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1]^2$ was simulated. In every simulation, the number of colleges (m) was 100 and every college had a capacity of $q = 5$. The number of students was parametrized: $n = \lambda q m$, where λ is the *market balance*, that is, the number of students per seat. Different values of λ allows us to see how the aggregate degree of competition between students for seats affects the results. We used the values $\lambda \in \{0.5, 1.0, 1.5\}$.

4.3.1 IDAM versus DA

Figure 4.1 shows, for each combination of α and β and λ, two values. In the first row of tables, we show the number of steps that it took for the unbounded IDAM to produce an outcome. That is, assuming that the time between each step of IDAM is set to be fixed, it represents the total amount of time that it takes for it to end and produce the student-optimal stable outcome. The second row shows the maximum of how far in each student's preferences the DA procedure had to go before producing the final outcome. That value, therefore, is the shortest length of the ranking for DA in that problem which can guarantee that truth-telling is a Nash equilibrium which yields the student-optimal stable outcome.

Some facts stand out. First, that the value of λ (the market balance) have significant qualitative impact in the outcomes. When $\lambda = 0.5$, both variables present a similar behavior: they increase with α and do not change much with β. The fact that the number of steps in IDAM (and similarly the maximum rank in DA) increases with the correlation of preferences is natural: as preferences become more correlated, many students follow a similar order of applications, with a small number of them being matched to their final allocation at each step.

Some of the most noticeable results are observed when $\lambda = 1$, that is, the number of students equals the number of colleges. Since all colleges and students are mutually acceptable, every student will be matched to a college and, therefore, the mechanisms will just determine the specific match of each student. What we will see is that the performance of IDAM and DA are almost complementary: IDAM performs better where DA is worse, and vice-versa. More specifically, when both preferences and grades have low correlation values, IDAM performs especially badly, and DA performs best. The reason for the bad performance of IDAM is that these scenarios are more prone to so-called rejection chains, in which one student applies to a college, which leads to the rejection of another student, who then applies and is tentatively accepted by another college, leading to a further rejection, etc. Since grades are not very correlated, the fact that a student was rejected at one college does not correlate with her being rejected at the next in her preferences, which increases the likelihood of those cycles. When preferences are more correlated, on the other hand, the number of students applying to a college is higher, and that competition makes it less likely that some student will later on remove one who was tentatively matched there. As a result, when α is high, this problem is reduced.

The performance of DA, on the other hand, has a different nature. When the values of α are low, there is overall less competition between the students for each college. As a result, it

is possible to satisfy students' preferences to a great extent, matching most of them among their most preferred colleges. When the value of β is high, though, some students have low grades in all colleges, and will therefore end up matched to colleges with seats left empty by other students' choices. As a result, these students will be matched to colleges lower in their preferences, and DA will also perform worse.

Overall, therefore, when it comes to the trade-off between rank length and number of steps when markets are balanced, IDAM and DA excel in complementary scenarios. When the correlation of preferences and grades are low, the execution of IDAM is extended for a large number of steps due to a small number of students following rejection chains, whereas in the other scenarios IDAM converges in a small number of steps, especially when grades are correlated.

When $\lambda = 1.5$, however, we see that while DA performs almost equally bad in most configurations, the effect that the rejection cycles have in the low correlation of preferences and grades is almost entirely eliminated by the increase in competition between students for the seats.[16] While this increase reduces the likelihood that a student who is rejected from a college is accepted in the next one, it doesn't change the fact that some students will end up matched to less desirable ones. For the intended application of large-scale national college admissions that use national exams, IDAM therefore presents its highest relative advantage: competition is high, grades are highly correlated, but preferences are less correlated, due to field and geographic preferences.

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[16]In the Appendix we show that when the ratio of students per seat is substantially higher, with $\lambda = 5.0$, this difference is even stronger, with IDAM using a smaller number of steps while DA still needs a full ranking to be submitted.
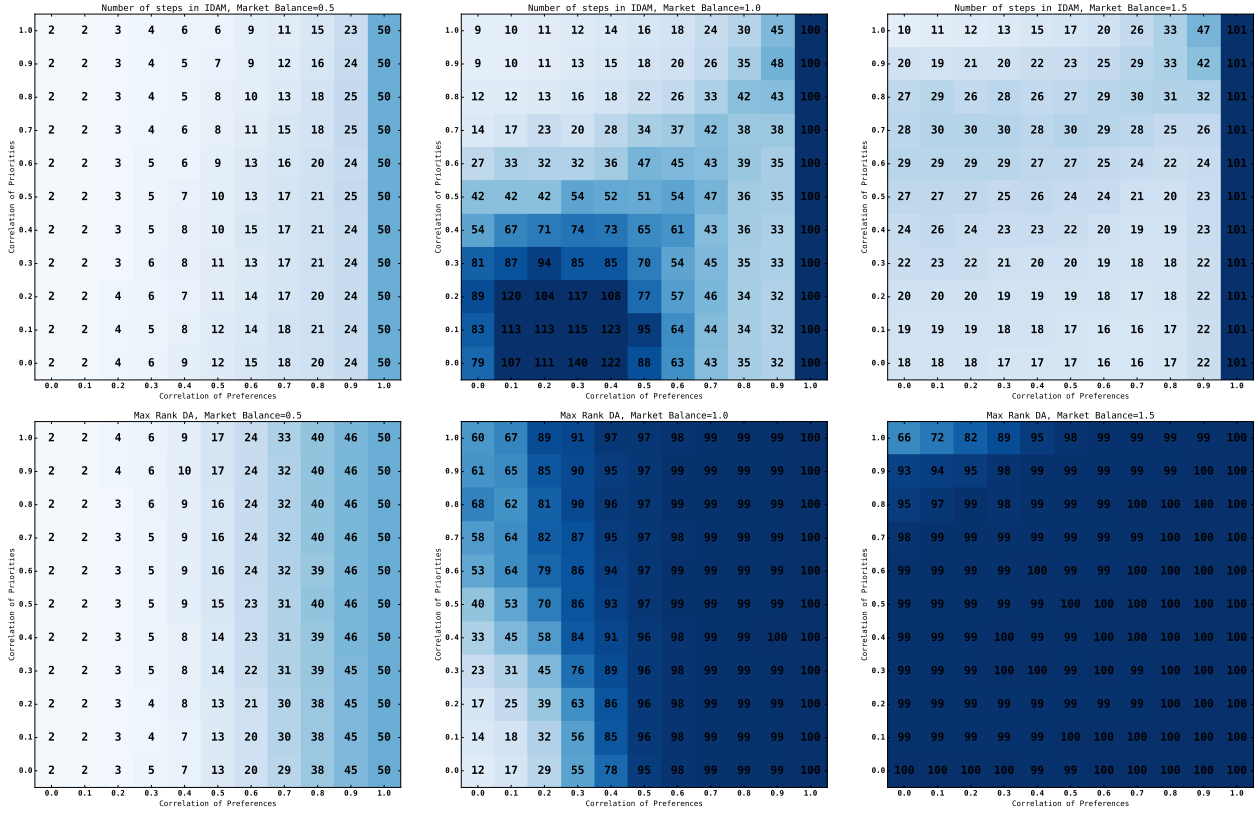
Figure 4.1: Number of steps in IDAM vs maximum rank in DA

4.3.2 IDAM+DA mechanism

In this simulation, we tested the IDAM+DA mechanism for different values of k. The simulations that we performed evaluate the following question: for a given value of k, what is the necessary length of the rank-ordered lists that students straightforwardly submit in step $k+1$ such that the outcome will still be the student-optimal stable outcome? By comparing those values with the length of the ranking necessary for DA to produce the same outcome, we are able to see how each step running IDAM reduces the length of the ranking that is necessary at the end, since DA is equivalent to the case where $k = 0$.

Figure 4.2 shows, for different values of β and λ, the average length of the ranking that is necessary for the mechanism to produce the student-optimal stable matching as the value of α varies. That is, it shows the average ranking of the worst matching between all students, *among the options given in the final menus.* The lines at the top represent the maximum ranking of a student used while running DA. The next line shows that value after running one step of IDAM. Especially when the market balance is larger ($\lambda \in \{1.0, 1.5\}$), the effect of

one single step may be quite large.

The intuition behind the reason why these initial steps have a large impact on the size of the ranking needed is the following. The length of the ranking necessary for producing the student-optimal stable matching is derived from the worst outcome among all students. These students are therefore only accepted in a deferred acceptance algorithm, after trying many other colleges. That is, those students likely have lower grades and are the ones who will see a larger reduction in the number of colleges available to them after the initial steps of the mechanism. In other words, these initial steps have a stronger effect on the students who have a *de facto* smaller set of options, given their grades and the preferences of higher-grade students. And these students are reasons why DA needs to allow for longer rankings.

These simulations indicate that a combination of some steps of IDAM followed by a ranking over the remaining options constitute a viable alternative for use in national college admission processes. It combines the desirable fact that the number of steps is fixed with the possibility that students, when asked for rankings over multiple options, are able to focus their analysis on a smaller set of options. Finally, these gains are stronger precisely in configurations present in these processes: lower correlation of preferences and higher correlation of grades.
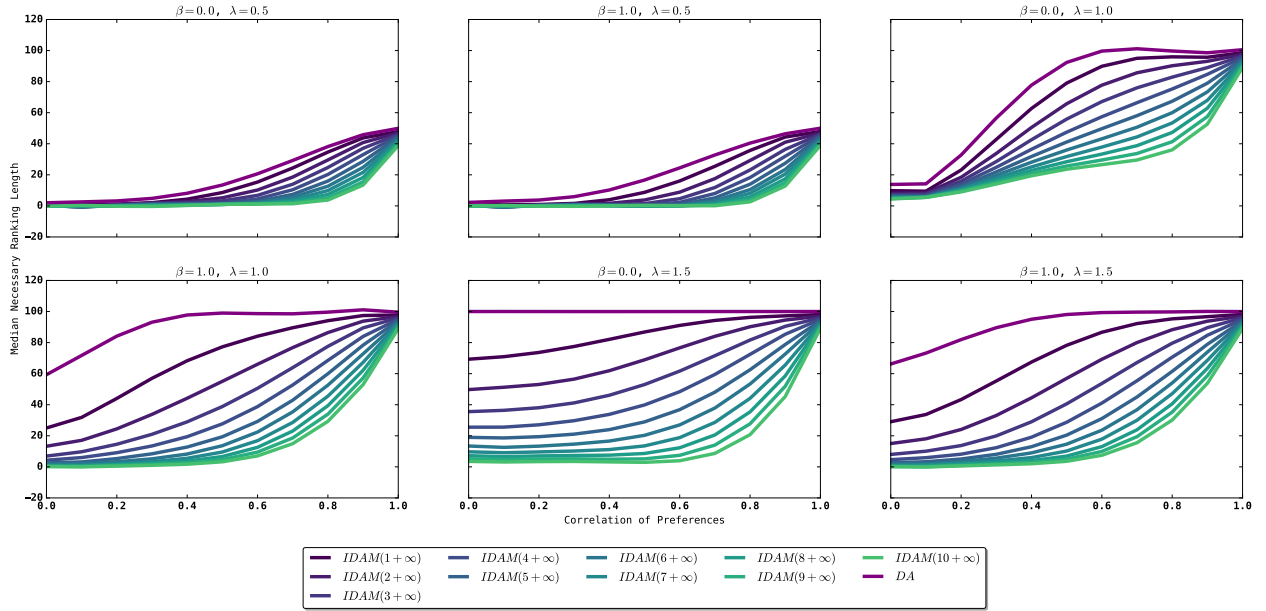
Figure 4.2: IDAM+DA : Minimum ranking length in DA

5 Conclusion

In this paper we analyze, both from an empirical and theoretical perspectives, the use of iterative mechanisms to produce stable allocations in one-sided matching markets. We show that a large real-life application of this family of procedures, the SISU mechanism, presents theoretical shortcomings that significantly reduce the benefits that could be obtained by these mechanisms, and are subject to manipulation via cutoffs. Empirical and anecdotal evidences show that these problems are not purely theoretical and may be affecting students' outcomes.

By introducing some modifications and generalizations to the idea behind the SISU mechanism, we introduced the GIDAM and IDAM mechanisms, which produce stable outcomes in equilibrium. By combining some iterative steps with a final request for a ranking over the remaining options, the IDAM+DA mechanism is an instance of GIDAM which uses the fact that a few steps of IDAM may clear a large proportion of the market, providing students with better information on the set of colleges that may actually accept them before asking for their preferences.

We believe that there is still many paths to follow ahead in the subject of iterative stable mechanisms. One of them, for example, is to use information that the policymaker may have about students' preferences, and optimize the mechanism accordingly. For example, if it is known that a large proportion of the students will have a certain college high in their preferences, the "adaptive" IDAM mechanism could start with a higher initial value for the cut-off at that college, and the stable matching would still be reached, in this case with a high probability.

Another related question would be the design of optimal menus which minimize the amount of information requested from the students, based on the known grades distribution. When grades are common, for example, the IDAM mechanism may obtain information on the preference that low-grade students have over "top" colleges, but if high-grade students are asked for their preferences earlier, it would not be necessary for this information to be revealed.

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A Appendix

A.1 Formal definitions

A.1.1 The generalized iterative deferred acceptance mechanism (GIDAM)

For any $I \subseteq X$, define $\mathcal{A}_c(I) \equiv \{x \in X \backslash I : x \in f_c(I \cup \{x\})\}$ and $\mathcal{A}_c^s(I) \equiv \mathcal{A}_c(I) \cap Y_s$. We denote by $\mathcal{A}_c^s(I)$ the set of **available contracts for** s **at** c **under** I. The interpretation of this is simple: a contract is available for student s at c under I if college c would choose to accept that contract while holding the set of contracts I.

Consider a college matching with contracts market $\langle S, C, T, X, P_S, F_C \rangle$, a maximum number of steps $T^{Max} \in \mathbb{N} \cup \{\infty\}$, a **maximum rank function** $\pi : \mathbb{Z}^+ \to \mathbb{N} \cup \{\infty\}$, a **public signal set** Θ, and a **public signal function** $\zeta : 2^X \to \Theta$. The **generalized iterative deferred acceptance mechanism** (**GIDAM**) proceeds as follows:

- **Step** $t = 0$: Let $\mathcal{L}^0 = S$, $S^0 = \varnothing$, and for every $c \in C$, $A^0(c) = \varnothing$. Make $\zeta(\varnothing)$ public. [17]

- **Step** $0 < t \leqslant T^{Max}$:

  - (a) Let $S^t \equiv \{s \in \mathcal{L}^{t-1} | \nexists x \in X_s, c \in C : x \in f_c(A^{t-1}(c))\}$. If $\pi(t) \neq \infty$, for every $s \in S$, let the menu of contracts presented to s be $\psi^t(s) \equiv \bigcup_{c \in C} \mathcal{A}_c^s(A^{t-1}(c)) \cup \{\varnothing\}$ if $s \in S^t$ and $\psi^t(s) = \varnothing$ otherwise. If $\pi(t) = \infty$, for every $s \in S$, let the menu of contracts presented to s be $\psi^t(s) \equiv \bigcup_{c \in C} \mathcal{A}_c^s(A^{t-1}(c)) \cup \{\varnothing\}$.

  - (b) If $\pi(t) = \infty$, request each student $s \in \mathcal{L}^{t-1}$ to submit a ranking of any size of elements in $\psi^t(s)$. If $\pi(t) \neq \infty$, request each student $s \in S^t$ to submit a ranking with at most $\pi(t)$ elements in $\psi^t(s)$. Let, for every student s', $P_{s'}^t$ be the ranking submitted. For every student s'' such that $\psi^t(s'') = \varnothing$, let $P_{s''}^t = P_{s''}^{t-1}$ and, for all $c \in C$, $B^0(c) \equiv A^{t-1}(c)$. Start with $\tau = 0$ and let $\mathcal{L}^t = \mathcal{L}^{t-1}$.

    * Sub-step $\tau \geqslant 0$: Some student s in $\mathcal{L}^{t-1}$, who does not have a contract being held by any college, proposes her most-preferred contract with respect to P_s^t which has not yet been rejected, x. If $x = \varnothing$, remove s from $\mathcal{L}^t$ and from further consideration. Otherwise, college $c(x)$ *holds* x if $x \in \mathcal{A}_c(B^\tau)$, and *rejects* x if $x \notin \mathcal{A}_c(B^\tau)$. Let $B^{\tau+1}(c) = B^\tau(c) \cup \{x\}$ and for all $c' \neq c$, $B^{\tau+1}(c') = B^\tau(c')$.

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[17]Notation clarification: $\mathcal{L}^t$ is the set of students who are still active at the begining of step t, and S^t is the set of students who are active and do not have any contract being held by a college at the beginning of that step.

         * Repeat the process above until no student is able to propose a new contract. Let τ^* be the last step into that process.

    – (c) For each college c, let $A^t(c) = B^{\tau^*}(c)$.

    – (d) If for every $c \in C$ it is the case that $A^t(c) = A^{t-1}(c)$, stop the procedure.

    – (e) Otherwise, make $\zeta\left(\bigcup_{c \in C} A^t(c)\right)$ public, and proceed to the next step.

- Denote by T^* the last step executed in the procedure. Let $X^* = \bigcup_{c \in C} f_c\left(A^{T^*}(c)\right)$. X^* is the outcome of the GIDAMprocedure.

A.1.2 Extensive-form game formulations and equilibrium concept

Fix a set of colleges C and their choice functions F_C. The extensive game form G induced by the GIDAM mechanism is a tuple $(S, H, \Phi, P, O, \xi, \pi)$ consisting of:

- A finite set of players $S = \{s_1, \ldots\}$.

- A finite set of actions $A = \{a_1, \ldots\}$.

- A list of preferences over random outcomes $P = (P_{s_1}, \ldots)$.

- A maximum rank function $\pi : \mathbb{N} \to \mathbb{N}$.

- A set of finite histories H, which are a sequence of actions with the property that if $(a_i)_{i=1}^k \in H$, then for all $\ell < k$, $(a_i)_{i=1}^\ell \in H$. The null history, $h_\varnothing$ is also in H.

- At history h_0, nature draws the value of (X, P) from a joint distribution ξ, and each student s observes the realization of X and of P_s. The distribution ξ is common knowledge.

- Let Z be the set of terminal histories, that is, if $h \in Z$ where $h = (a_i)_{i=1}^k$, then there is no $h' \in H$, with $h' = (a_i')_{i=1}^\ell$ where $\ell > k$ and for all $i \leqslant k$, $a_i = a_i'$. Then $(a_i)_{i=1}^k \in Z \implies k \bmod n = 0$.

- Φ is a player function. $\Phi : H \backslash Z \to S$.[18] There exists an ordering of the players $(s_1, \ldots, s_n)$ such that, for all $h \in H$ such that $|h| \leqslant n$, $\Phi(h) = s_{|h|}$.[19]

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[18]For simplicity, we only allow one player per history. This is without any loss of generality.

[19]That is, the first n actions consist of player s_1 playing first, s_2 second, and etc.

– Let $(a_i)_{i=1}^{k} \in H$, where $k \geqslant 1$. If $(a_i)_{i=1}^{k+n} \in H$, then $\Phi\left((a_i)_{i=1}^{k}\right) = \Phi\left((a_i)_{i=1}^{k+n}\right)$.[20]

- For each student s, $\mathcal{I}_s$ is a partition of $h : \Phi(h) = s$. Define $\zeta\left((a_i)_{i=1}^{k}\right)$ as the list of public signals that result from the sequence of actions in $(a_i)_{i=1}^{k}$ and $\psi_s\left((a_i)_{i=1}^{k}\right)$ as the sequence of menus of contracts presented to student s after that same sequence of actions. Define $H_\ell^t \equiv \left\{(a_i)_{i=1}^{k} \in H : k \mod n = \ell \text{ and } k \div n = t - 1\right\}$, and let $h, h' \in H_\ell^t$. The histories $h = (a_i)_{i=1}^{k}$ and $h' = (a_i')_{i=1}^{k}$ belong to the same member of the partition $\mathcal{I}_{s_\ell}$ if and only if:

  – $|h| \mod n = |h'| \mod n$,

  – $\zeta(h) = \zeta(h')$,

  – $\psi_{s_\ell}(h) = \psi_{s_\ell}(h')$,

  – $a_i = a_i'$ for all i such that $i \mod n = \ell$.[21]

- $A(h)$ are the actions available at $h \in H$. For every $h_i \in H_\ell^t$, the set of actions depend on whether, given the history of actions until step t of the GIDAM mechanism, student $s = \Phi(h_i)$ is offered a non-empty set of contracts, in which case $A(h_i)$ is the set of the ordered list of the contracts in $\psi^t(s)$ with at most $\pi(t)$ elements, or not, in which case we denote $A(h_i) = \{\diamondsuit\}$, where $\diamondsuit$ is simply a placeholder for an action when no action is requested from the student. We abuse notation and denote, for any $I_i \in \mathcal{I}_s$, $A(I_i)$ to be $A(h_i)$ for any $h_i \in I_i$ (remember that by definition all histories in I_i have the same set of actions associated with them).

- A strategy for player s is a function $\sigma_s(\cdot)$ that assigns an action in $A(I_i)$ to each information set $I_i \in \mathcal{I}_s$.[22]

- The outcome function O assigns, to each strategy profile $\sigma = (\sigma_{s_1}, \ldots, \sigma_{s_n})$, a random outcome that results from following the histories that result from following those strategies in the GIDAM mechanism, given each realization of X and P.

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[20]This, combined with the previous item and the condition on terminal histories, implies that every player plays every n actions once.

[21]That is, two histories belong to the same set of the partition if the student's preferences are the same and the history of publicized sets of acceptable contracts was the same, and the actions taken by that player were also the same.

[22]We restrict our analysis to pure strategies.

Since our solution concept will demand that students' strategies are rational at all possible information sets, we will need to consider how students' strategies act at each subgame. We first define a subgame:

Definition 6. A **subgame** of the game G at non-terminal history $h = (a_i)_{i=1}^k$, for $h \in H \backslash Z$ is a game $[G|_h = (S|_h, H|_h, \Phi|_h, P|_h, O)] < G|_h = (S|_h, H|_h, \Phi|_h, P|_h, \pi, O) >$ (we may also abuse notation and let $G|_{I_i} = G|_h$ when $h \in I_i$) where:

- $H|_h = \left\{ h' = (a'_i)_{i=k}^l \text{ where } l \geqslant k \text{ and } (a_1, \ldots, a_{k-1}, a'_k, \ldots, a'_l) \in H \right\}$

- $S|_h = \{ s \in S : \Phi(h') = s \text{ for some } h' \in H|_h \backslash Z \}$

- $\Phi|_h : H|_h \rightarrow S|_h$ such that for all $h' \in H|_h$, where $h' = (a'_i)_{i=k}^l$, and $\Phi|_h(h') = \Phi(a_1, \ldots, a_{k-1}, a'_k, \ldots, a'_l)$

- For each $s \in S|_h$, $P_s|_h$ satisfies, for all $h', h'' \in H|_h$:

$$ h' \, P_s|_h \, h'' \iff (a_1, \ldots, a_{k-1}, a'_k, \ldots, a'_l) \, P_s \, (a_1, \ldots, a_{k-1}, a''_k, \ldots, a''_l) $$

- The weak preference $R_s|_h$ is defined accordingly.

Finally, let $\sigma|_h = (\sigma_{s_1}|_h, \ldots, \sigma_{s_n}|_h)$ be the strategy profile σ restricted to the subgame $G|_h$. We can define analogously a subgame in terms of an information set instead of a single history. Let t^∞ be the highest value of t such that $\pi(t) = \infty$. We will consider situations in which students present straightforward behavior. Therefore, we can define a straightforward strategy accordingly:

Definition 7. A strategy σ_s of student $s \in S$ is **straightforward with respect to P^*** if for every t, $h_s^t \in H_s^t$ and $\sigma_s(h_s^t | z(s), P^*) = \Diamond$ if $A(h_s^t) \neq \{\Diamond\}$. Otherwise, $\sigma_s(h_s^t | z(s), P^*)$ consists of the π^* most preferred contracts in $A(h_s^t)$, ordered according to P^*, where $\pi^* \leqslant \pi(t)$, and $\pi^* = |A(h_s^t)|$ when $t = t^\infty$.

Let A and B be two random outcomes. We denote by $>_s$ the first-order stochastic dominance relation under P_s. That is, $A >_s B$ if for all $v \in C \cup \{s\}$, $Pr\{A(s) = v' | v' R_s v\} \geqslant Pr\{B(s) = v' | v' R_s v\}$. A **belief system** ω is a collection of probability measures, one for each information set. Moreover, denote by $O_\omega(\sigma)_G$ the random outcome induced by the strategy profile σ and belief system ω in game G.

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Definition 8. A strategy profile σ together with a belief system ω is an **ordinal perfect Bayesian equilibrium (OPBE)** of a game G if:

(i)   For every $I_i \in \mathcal{I}_s$ and $s \in S|_{I_i}$, $O_\omega \left(\sigma_s|_h, \sigma_{-s}|_h \right)_{G|_{I_i}} \succ_s O_\omega \left(\sigma'_s|_h, \sigma_{-s}|_h \right)_{G|_{I_i}}$

(ii)   Let $Pr\left(h|\sigma \right)$ be the probability that history h is reached, given σ. The belief system satisfies the following property, for any information set I_i that is reached with positive probability, and $h \in I_i$: $\omega\left(h \right) = \frac{Pr(h|\sigma)}{\sum_{h' \in I_i} Pr(h'|\sigma)}$.

A.2   Other properties of the SISU mechanism

One interesting property of the IDAM mechanism, is that although the combination of strategies that students may use is much richer than that of straightforward strategies, the sequence of interactions that the students have with the mechanism cannot be distinguished from interactions that result from all students following straightforward strategies.

Lemma 2. *Fix a realization of P_S and z, let $\sigma = \left(\sigma_{s_1}, \ldots, \sigma_{s_n} \right)$ be a strategy profile and h a history that results from that strategy profile up to some step t. There is at least one strategy profile σ^*, where every student follows a straightforward strategy with respect to some preference profile P^*, which also results in history h up to step t.*

Lemma 2 can be derived from intermediate results of the proof of Theorem 1, and is ommitted. This result does not hold for the SISU mechanism, however.

Remark 4. There are sequences of actions that students may take under the SISU mechanism that cannot be produced by any profile of straightforward strategies.

To see why Remark 4 is true, consider a student who has the highest grade in colleges c_1 and c_2, and on day 1 chooses college c_1, on day 2 chooses c_2, and day 3 chooses c_1 again. This sequence of actions is not possible under the IDAM mechanism, cannot be the result of a straightforward strategy (since in all steps both colleges are available to her), but can take place under the SISU mechanism.

As the proposition below shows, this results on the fact that all students following the straightforward strategy is not necessarily an equilibrium under the SISU mechanism.

Proposition 6. *The strategy profile in which all strategies are straightforward may not constitute a Nash Equilibrium of the game induced by the SISU mechanism, for any value of T^{Max}.*

Proof. Consider the set of students $S = \{s_1, s_2, s_3, s_4\}$ and of colleges $C = \{c_1, c_2, c_3, c_4\}$, each with capacity $q_i = 1$ and minimum score zero. Students' preferences are as follows:[23]

$$P_{s_1} : c_1 \ c_4 \ c_3 \ c_2$$
$$P_{s_2} : c_1 \ c_2 \ c_3 \ c_4$$
$$P_{s_3} : c_2 \ c_3 \ c_1 \ c_4$$
$$P_{s_4} : c_3 \ c_1 \ c_2 \ c_4$$

Students' exam grades at the colleges are as follows:

	c_1	c_2	c_3	c_4
s_1	300	200	100	100
s_2	200	400	200	200
s_3	100	300	400	300
s_4	400	100	300	400

If all students follow the straightforward strategy, students' choices over time are the following:

	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
s_1	c_1	c_1	c_1	c_1	c_4
s_2	c_1	c_2	c_2	c_2	c_2
s_3	c_2	c_2	c_3	c_3	c_3
s_4	c_3	c_3	c_3	c_1	c_1

Notice that in step $t = 5$, all students' choices are accommodated. Therefore, even if the SISU mechanism runs for more than five steps, students following the straightforward strategy would keep their choices. Therefore, the matching produced would be the following, which is the student-optimal stable matching:

$$\mu = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ s_4 & s_2 & s_3 & s_1 \end{pmatrix}$$

Therefore, for any $T^{Max} \geqslant 5$, the outcome of the SISU mechanism, when students follow the straightforward strategy, is μ. Without loss of generality, let $T^{Max} = 5$. Suppose that student s_1, instead of choosing c_1 in the first and second steps, chooses c_4 and then, from

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[23]This example is based on Example 1 in Kesten [2010].

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step $t = 3$ on, follows the straightforward strategy. Then students' choices over time will be as follows:

	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
s_1	c_4	c_4	c_1	c_1	c_1
s_2	c_1	c_1	c_1	c_2	c_2
s_3	c_2	c_2	c_2	c_2	c_3
s_4	c_3	c_3	c_3	c_3	c_3

$$\mu' = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ s_1 & s_2 & s_3 & \varnothing \end{pmatrix}$$

Student s_1, therefore, is matched to her most preferred college, as opposed to the case when she follows the straightforward strategy. $\qquad\square$

The proof of Proposition 6 is based on the fact that when students follow straightforward strategies (as well as during the execution of the DA algorithm), there may be *rejection cycles*. That is, in some step a student s chooses a college which makes another student, who becomes tentatively rejected, change her choice, and so on, until some other student ends up making it impossible for s to be matched to the college she chose in the first place. When that is the case, one thing that s could do is "postpone" her choice, so that the rejection cycle does not reach her before step T^{Max}.

A.3 The SISU 2016 procedure

The procedure used during the 2016 version of the SISU selection process has two differences when comparing with the 2010 version, described in section 2: the presence of affirmative action quotas and the fact that, instead of choosing only one program in each step, students were able to choose a first and a second option.

A.3.1 Affirmative action quotas

Due to a federal affirmative action law, many of the programs offered in the SISU have their seats split into five sets: Q_{MI}^c, Q_{mI}^c, Q_{Mi}^c, Q_{mi}^c and Q_-^c, with capacities q_{MI}^c, q_{mI}^c, q_{Mi}^c, q_{mi}^c and q_-^c, respectively. A student may apply to only one among these in a program, but eligibility to apply to each of those depend on her status as a low-income student, as a student who is a racial minority and whether she studied in a public high school:

- Only candidates who studied in a public high school, belong to a racial minority, and provide evidence of belonging to a low-income household may apply to a seat in Q_{MI}^c,

- Only candidates who studied in a public high school and provide evidence of belonging to a low-income household may apply to a seat in Q_{mI}^c,

- Only candidates who studied in a public high school and belong to a racial minority may apply to a seat in Q_{Mi}^c,

- Only candidates who studied in a public high school may apply to a seat in Q_{mi}^c,

- Any candidate may apply to seats in Q_-^c.

The cutoff values that are published by the SISU mechanism are calculated, for each set of seats in each program, exactly the same as described in section 2. The cutoff for the seats in Q_{mI} in step t is the q_{mI}th highest exam grade in the program among those who applied to the set of seats Q_{mI}. It is easy to see, therefore, that if cutoffs go down, students who would have ultimately been eligible for those seats may believe that they would now not be accepted into that program.

A.3.2 First and second choice

Differently from the procedure used in 2010, in 2016, in each step, students were asked to indicate a first and second choice among the available sets of programs.[24] The cutoff values calculated at the end of each step were based on the execution of the following algorithm:

1. Consider all students' first and second choice as being their preferences over C. That is, if student s submitted c_1 as her first choice and c_2 as her second choice, her full preference are considered as being $c_1 P_s c_2 P_s s$.

2. Using programs' preferences responsive to exam scores, let μ_t^C be the outcome of the **college-proposing deferred acceptance** algorithm.

3. For each program c such that $\left| \mu_t^C(c) \right| < q_c$, let the cutoff for that step ζ_c^t be $\underline{z}_c$, that is, the minimum exam grade for acceptance at c.

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[24]More precisely, the students could choose two programs, first and second choice, and only one option among the options sets of seats described in section A.3.1 in each. For example, the first choice could be the seats reserved for low-income minorities in program c_1 and the second choice would be the seats reserved for low-income non-minorities in program c_2. Since for the purpose of the analysis of the effect of the availability of these two choices this fact is not relevant, we will consider simply the students' choices over programs.

4. For each program c such that $\left|\mu_t^C(c)\right| = q_c$, let the cutoff for that step ζ_c^t be $\min_{s \in \mu_t^C(c)} z_c(s)$, that is, the lowest exam grade at c among those in $\mu_t^C(c)$.

It is easy to see that the procedure in 2010 is the same as the one described above with the difference being that instead of using two choices as the student's preference one considers instead a preference in which a student considers only one college acceptable. Our objective in this section is to show that the problems associated with the fact that cutoff grades go down are also present in the version of the mechanism used in 2016.

Since students can only submit two colleges, it is still the case that the students have to carefully consider which one to submit, since she may end up not matched to any college. What is left to evaluate, however, is whether a student would always be able to be accepted at a college which has a cutoff value lower than her grade at that college.

Consider an exam-based college matching market $\langle S, C, q, P_S, z, \underline{Z} \rangle$, let $P_S^* = (P_s^*, P_{-s})$ be a preference profile where P_s^* differs from P_s in that college $c^* \in C$ is deemed as unacceptable for s and let μ^C be the college-optimal stable matching for the exam-based college matching market $\langle S, C, q, P_S^*, z, \underline{Z} \rangle$. Suppose that μ^C is blocked, with respect to P_S, by s and c^*. We want to show that the college-optimal stable matching μ^{C*} for the market $\langle S, C, q, P_S^{**}, z, \underline{Z} \rangle$, where $P_S^{**} = (P_s^{**}, P_{-s})$ and P_s^{**} is the preference for s in which only college c^* is acceptable, is such that $\mu^{C*}(s) = c^*$.

To see that this is the case, consider the college-proposing deferred acceptance procedure for the market $\langle S, C, q, P_S^*, z, \underline{Z} \rangle$ and college c^*. At each step of the deferred acceptance procedure, college c^* proposes to the top q_{c^*} students, who had not yet rejected c^*, with respect to z_{c^*}. Let T^* be the number of steps in the algorithm until the matching μ^C is produced. The set $\mu^C(c^*)$ consists of the top students in S, with respect to z_{c^*}, who did not reject c^* at some step. Since μ^C is blocked by s and c^*, either $\left|\mu^C(c^*)\right| < q_{c^*}$ or there is a student $s' \in \mu^C(c^*)$ such that $z_{c^*}(s) > z_{c^*}(s')$.

Suppose that the statement is false, that is, that the college-optimal stable matching μ^{C*} under the market $\langle S, C, q, P_S^{**}, z, \underline{Z} \rangle$ is such that $s \notin \mu^{C*}(c^*)$. Since offers made by colleges during the college-proposing DA are not withdrawn, it must be that college c^* does not make an offer to s, implying that college c^* has under μ^{C*} a "more preferred" set of students than under μ^C. Now consider the deferred acceptance steps in $\langle S, C, q, P_S^{**}, z, \underline{Z} \rangle$ compared to those in $\langle S, C, q, P_S^*, z, \underline{Z} \rangle$, and consider the first step at which the proposals and rejections are different. These differences can only happen if student s rejects colleges considered acceptable under P_s^* but unacceptable under P_s^{**}. That is, they come from the fact that student s will reject a set of colleges weakly larger than under P_s^*. This implies

that those colleges will make offers that are further down in their ranking. Those further rejections will also, at each step, weakly increase the set of students who reject offers from colleges. That is, in terms of colleges' "preferences," they are weakly worse off. But this contradicts the assumption that $s \notin \mu^{C*}(c^*)$ and s blocking μ^C with c^*, since it would be necessary for c^* to obtain a more preferred set of students under μ^{C*} in order not to make an offer to s.

Therefore, the result above shows that if a program has a cutoff value that is below a student's grade in that program, the student would be able to get accepted at that college by modifying her preference, putting that program as her top choice. Similarly, if the cutoff is higher than that student's grade, that student would never receive an offer from that college during the deferred acceptance procedure.

We can therefore conclude that the problems associated with the possibility of a reduction in the cutoff values described in section 2 are also present in the 2016 version of the SISU mechanism.

A.4 Proofs

Proposition 2

Proof. For this proof we consider an exam-based college matching market and an IDAM mechanism with $\pi(t) = 1$ for all t. Consider the set of students $S = \{s_1, s_2, s_3\}$ and of colleges $C = \{c_1, c_2, c_3\}$, each with capacity $q_i = 1$. Student s_1, who will be the player to whom we will show no dominant strategy exists, has preferences $c_1 P_{s_1} c_2 P_{s_1} c_3$, and students' exam grades at those colleges are as follows:

	c_1	c_2	c_3
s_1	100	100	100
s_2	200	200	200
s_3	300	300	300

Suppose now that, conditional on the realized preferences and grades of student s_1, student s_3 follows a straightforward strategy with respect to the preference $c_3 P^3 c_2 P^3 c_1$. Notice that we are not stating those are the preferences of student s_3, we are simply assuming that she will follow the *straightforward strategy with respect to P^3*. Next, we consider two strategies for student s_2 and show that no strategy is a common best response for these two possibilities.

Scenario 1

Suppose that student s_2's strategy is the following: in $t = 1$, choose c_3. If at some later point s_2 is asked again to make a choice, she will choose the college with the highest cutoff value at that step among the options available. In case of ties, she will choose the college with the lowest index number (for example, the index number of c_2 is 2). We will show that, given s_2 and s_3's strategies, the best response involves first choosing c_2. The sequence of steps will be as follows:

Step 1: Student s_1 applies to c_2. Students s_2 and s_3 apply to c_3. Student s_2 is rejected. Cutoffs $\left(\zeta_{c_1}^1, \zeta_{c_2}^1, \zeta_{c_3}^1\right)$ are $(0, 100, 300)$.

Step 2: Since $\zeta_{c_2}^1$ is the highest cutoff among the colleges offered to s_2, student s_2 applies to c_2. Student s_1 is rejected. Cutoffs $\left(\zeta_{c_1}^2, \zeta_{c_2}^2, \zeta_{c_3}^2\right)$ are $(0, 200, 300)$.

Step 3: Student s_1 is left with two options: choose c_1 or s. If she chooses s she will remain unmatched. If she applies to c_1, she will be accepted. Final cutoffs $\left(\zeta_{c_1}^3, \zeta_{c_2}^3, \zeta_{c_3}^3\right)$ would then be $(100, 200, 300)$ and the outcome would be the matching μ' as follows:

$$\mu = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_1 & s_2 & s_3 \end{pmatrix}$$

Student s_1 can therefore be matched to her most preferred college by first choosing c_2. We now show that by choosing first c_1 or c_3, s_1 will always be matched to a strictly inferior college. First, let her choose c_1 first:

Step 1: Student s_1 applies to c_1. Students s_2 and s_3 apply to c_3. Student s_2 is rejected. Cutoffs $\left(\zeta_{c_1}^1, \zeta_{c_2}^1, \zeta_{c_3}^1\right)$ are $(100, 0, 300)$.

Step 2: Since $\zeta_{c_1}^1$ is the highest cutoff among the colleges offered to s_2, student s_2 applies to c_1. Student s_1 is rejected. Cutoffs $\left(\zeta_{c_1}^2, \zeta_{c_2}^2, \zeta_{c_3}^2\right)$ are $(200, 0, 300)$.

Step 3: Student s_1 is left with two options: choose c_2 or s. If she chooses s she will remain unmatched. If she applies to c_1, she will be accepted. Final cutoffs $\left(\zeta_{c_1}^3, \zeta_{c_2}^3, \zeta_{c_3}^3\right)$ would then be $(200, 100, 300)$ and the outcome would be the matching μ' as follows:

$$\mu' = \begin{pmatrix} c_1 & c_2 & c_3 \\ s_2 & s_1 & s_3 \end{pmatrix}$$

If s_1 chooses c_3 first instead, the following will happen:

Step 1: Students s_1, s_2, and s_3 apply to c_3. Students s_1 and s_2 are rejected. Cutoffs $\left(\zeta_{c_1}^1, \zeta_{c_2}^1, \zeta_{c_3}^1\right)$ are $(0, 0, 300)$.

Step 2: Following her strategy and the fact that college c_1's index is lower than c_2, student s_2 applies to c_1. Student s_1 has three options: choose also c_1 and therefore be rejected and

left to choose between c_2 and s in step $t = 2$, choose c_2 or choose s. In all cases she will either end up remaining unmatched or matched to c_2.

Scenario 2

Now suppose that student s_2 follows a similar strategy to scenario 1, but where instead of applying to c_3 and then to the college with the highest cutoff value, she applies to the college with the *lowest* cutoff value, once again breaking ties based on the index of the college.[25] Following an exercise similar to the one above, it is easy to see that student s_1's strategies that involve choosing first c_2 or c_3 will lead her to either be unmatched or be matched to c_2, while choosing c_1 will match her to c_1, her most preferred college.

Since every best response strategy under scenario 1 is dominated by different strategies in scenario 2, we have shown that a student may not have a weakly dominant strategy of the game induced by the IDAM mechanism, and as a consequence also the GIDAM mechanism.

□

Proposition 3

Proof. Since students follow straightforward strategies, a student $s \in S^i$ will only apply to colleges that are not in C^i if the cutoffs at all colleges in C^i are above her exam grade. Moreover, since for every i the number of students who prefer any college in C^i to any college not in C^i is at least as big as the overall number of seats in these colleges, by the end of the execution of the IDAM mechanism all seats in those colleges will be occupied by students in S^i, and students in S^i who are not matched to colleges in C^i will be left unmatched (even though some of them may be tentatively accepted at some step during the execution of the mechanism). From the perspective of a student in S^i, therefore, a seat in a college in C^i which is being occupied by a student not in S^i is equivalent to an empty seat.

Consider any $i \in \{1, \ldots, k\}$ and let $q_1^i \leqslant q_2^i \leqslant \cdots \leqslant q_{|C^i|}^i$ be the ordered capacities of the colleges in C^i. We will denote by $\left\{ S_1^i, S_2^i, \ldots, S_{|C^i|}^i, S_-^i \right\}$ the partitioning of the students in S^i where S_1^i are the top q_1^i students in S^i in colleges C^i's preferences, S_2^i are the top q_2^i students after those in S_1^i in colleges C^i's preferences, etc. and S_-^i are the students in S^i below the top $\sum_{j=1}^{|C^i|} q_j^i$ students. By Proposition 1, when students follow straightforward strategies the

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[25]Although the strategies used in this proof for student s_2 may seem very arbitrary, they can be rationalized by two simple stories. Student s_2's strategy in scenario 1 is consistent with a student who knows that her top choice is c_3 but that has some uncertainty about which one between c_1 and c_2 is her second choice, and sees the cutoff grade as an indication of how competitive acceptance is at those colleges and therefore sees the perceived quality of those. The strategy in scenario 2 could be rationalized by a student who once again knows that her top choice is c_3 but that would otherwise prefer to go with a college with low-achieving peers, and uses the low cutoff as an indication of that fact.

final outcome of the GIDAM mechanism is the student-optimal stable matching. We will use the following Lemma:

Lemma 3. *Let all students present straightforward behavior with respect to the preference profile P and μ be the matching produced by the IDAM mechanism. If a student s blocks μ with some college c, then $\mu(s) = s$.*

Proof. If the IDAM mechanism is run for enough steps, Proposition 1 implies that μ is stable and therefore no student blocks μ with any college. Consider now the case in which the number of steps T^{Max} is smaller than that, and suppose that there is a student s and a college c where $cP_s\mu(s)$, $\mu(s) = c'$ and student s and college c block μ. Since $\mu(s) = c'$, then at some step $t^* \leqslant T^*$, s chose college c'. Since s and c block μ, it must be that $\zeta_c^{T^*} < z_c(s)$. By Corollary 3, $\zeta_c^{t^*} \leqslant \zeta_c^{T^*}$. Therefore, in step t^* both colleges c and c' were available to s but she chose c'. A contradiction with straightforward behavior with respect to P. $\qquad\square$

By Lemma 3, at any step in which there are blockings, those involve students who are not tentatively matched to any college. The number of students who are involved in a block is therefore maximized when the number of students tentatively accepted to a college in any step is minimal.

Consider now step $t = 1$. Since every college in C^i has at least q_1^i seats, every student in S_1^i will be accepted at any college in that step. There is one case in which all other students will be rejected, though: if all students in S^i choose the same college with capacity q_1^i in step $t = 1$. In that case, $|S^i| - q_1^i$ students in S^i will be tentatively unmatched by the end of step 1, and therefore if the GIDAM mechanism runs for just one step, that is, the maximum number of students in S^i who will be involved in blocking pairs. The same argument will follow at $t = 2$: given that the students in S_1^i are all matched to a college with capacity q_1^i, the number of students who are tentatively unmatched by step $t = 2$ is maximal when all the remaining students in S^i choose a college with capacity q_2^i.

If we consider all the colleges and students, this process will take place in parallel at each element of $S = \{S^1 \cup S^2 \cup \cdots \cup S^k\}$ and $C = \{C^1 \cup C^2 \cup \cdots \cup C^k\}$. That is, by the end of step 1, the maximum number of students involved in blocks in S^1 is $|S^1| - q_1^1$, in S^2 is $|S^2| - q_1^2$, etc. The result therefore extends to a maximum of $n - \sum_{j=1}^{k} \sum_{i=1}^{T} q_i^j$ students involved in blocks.

Finally, if we consider the maximum number of steps that it takes until the student-optimal stable matching is produced, we can ask about which preferences from the students minimize the number of students who are matched to their final allocation at each step.

That is, by minimizing the number of students matched to their final allocation we allow for the maximum number of students who can still make choices. Here it is easy to see that the preferences considered above, in which all students apply to the colleges in order of increasing capacity, is also the one that at each step matches the minimal number of students to their final allocation. The overall process will in that case end when the last set in $\left\{S^1_{|C^1|}, S^2_{|C^2|}, \ldots, S^k_{|C^k|}\right\}$ is matched to their final allocation. That will therefore be the one with the largest number of colleges. Thus, the maximum number of steps is $\max_i \{|C^i|\}$. $\quad\square$

Lemma 1

Proof. First, note that if $\psi^{t'}(s) = \varnothing$ then the statement is true. Suppose, for contradiction, that $\psi^{t'}(s) \neq \varnothing$ and the statement is false. Then there is a student $s \in S$, $0 \leqslant t \leqslant t' \leqslant T^*$ and a contract $x^* \in X$ such that $x^* \in \psi^{t'}(s)$ but $x^* \notin \psi^t(s)$. Since, for any $I' \subseteq X$, $\mathcal{A}_c(I) \subseteq I_c$, we can separate the violation of the lemma into the contracts available for a student from a single college. There is, therefore, a college c such that:

$$x^* \in \mathcal{A}^s_c\left(A^{t'}(c)\right) \text{ but } x^* \notin \mathcal{A}^s_c\left(A^t(c)\right)$$

By construction, for every $c \in C$, $A^t(c) \subseteq A^{t'}(c)$. Therefore:

$$x^* \in \mathcal{A}^s_c\left(A^{t'}(c) \cup \left[A^{t'}(c) \backslash A^t(c)\right]\right) \text{ but } x^* \notin \mathcal{A}^s_c\left(A^t(c)\right)$$

And given the definitions of $\mathcal{A}^s_c$ and $\mathcal{A}_c$:

$$x^* \in f_c\left(A^t(c) \cup \left[A^{t'}(c) \backslash A^t(c)\right] \cup \{x^*\}\right) \text{ but } x^* \notin f_c\left(A^t(c) \cup \{x^*\}\right)$$

We need to consider three cases: (i) $x^* \notin A^{t'}(c) \cup A^t(c)$, (ii) $x^* \in A^t(c)$ and (iii) $x^* \in A^{t'}(c) \backslash A^t(c)$. Cases (i) and (ii): In both cases, $x^* \notin \left[A^{t'}(c) \backslash A^t(c)\right]$. Since f_c satisfies IRC, $f_c\left(A^t(c) \cup \left[A^{t'}(c) \backslash A^t(c)\right] \cup \{x^*\}\right) = f_c\left(A^t(c) \cup \{x^*\}\right)$. But this contradicts $x^* \notin f_c\left(A^t(c) \cup \{x^*\}\right)$. Case (iii): Here we will use the following claim, which can easily be derived from the definition of unilateral substitutes:

If contracts are unilateral substitutes for college c under f_c, there does not exist contract $z \in X_s$ and sets of contracts $Y \subseteq X \backslash X_s$ and $I \subseteq X \backslash X_s$ such that $z \notin f_c(Y \cup \{z\})$ and $z \in f_c(Y \cup I \cup \{z\})$.

Denote by $I^* = \left[A^{t'}(c) \backslash A^t(c)\right] \backslash \{x^*\}$. Then:

$$x^* \in f_c\left(A^t(c) \cup I^* \cup \{x^*\}\right) \text{ but } x^* \notin f_c\left(A^t(c) \cup \{x^*\}\right)$$

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By IRC and the fact that f_c chooses only one contract per student:

$$x^* \in f_c\left(\left[A^t\left(c\right)\backslash X_s\right] \cup \left[I^*\backslash X_s\right] \cup \{x^*\}\right) \text{ but } x^* \notin f_c\left(A^t\left(c\right)\backslash X_s \cup \{x^*\}\right)$$

Following the claim above, this contradicts the assumption that f_c satisfies unilateral substitutes, finishing the proof. □

Proposition 1

Proof. First, note that given the description of the unbounded GIDAM mechanism and Lemma 1, every time a student is asked to submit a ranking, the set of contracts available under the GIDAMmechanism is weakly smaller. Moreover, for any t, t' and $s \in S$ such that $0 \leqslant t < t' \leqslant T^*$, $\psi^t\left(s\right) \neq \varnothing$ and $\psi^{t'}\left(s\right) \neq \varnothing$, it must be that the set of contracts in $\psi^{t'}\left(s\right)$ is a strict subset of $\psi^t\left(s\right)$, since at least the highest ranked contract submitted by student s in step t must have been rejected by step t'. Therefore, in every step the set $\psi^t\left(s\right)$ is strictly smaller for at least one student. Since X is finite, GIDAM will end and will produce an outcome after a finite number of steps.

Next, notice that regardless of which straightforward strategy students use, in all of them students will offer contracts following the order of their preference, perhaps only skipping those which would not be held by the college associated with the contract, and that the outcome will be produced when every student either chooses $\varnothing$, or has a contract held by a college or reaches the end of the last ranking submitted. By Hirata and Kasuya [2014], if the choice functions used by the colleges satisfy unilateral substitutes and IRC, this (cumulative offer) process will produce the student-optimal stable matching regardless of the order in which doctors are called to offer contracts, as long as the order in which each student offers her contracts follow their preferences over them. Different straightforward strategies may imply different orders in which students offer contracts, but not the fact that students follow their own preference until the end.[26] Therefore, the outcome of the GIDAM mechanism, for any profile of straightforward strategies, will always be the student-optimal stable matching. □

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[26]Technically speaking, under the cumulative order process students will always offer contracts following their preference, even those which would not be accepted by the college in the contract. Since choice functions satisfy IRC, however, this is equivalent to a process that simply skips those contracts that would not be accepted (and are, therefore, not part of the menus offered to the students under the GIDAM mechanism).

Theorem 1

Proof. We use the extensive game notation introduced in the Appendix A.1. Consider some history $h \in H$. Given other players' strategies σ_{-s}, the history that results from the strategy profile (σ_s, σ_{-s}) consists, as described in the definition of the GIDAMmechanism, of a series of steps in which each student has either only the action $\diamondsuit$ or some menu of options $\psi^t(s)$ and a maximum rank $\pi(t)$. Therefore, given our strategy profile and student s, we can write down a list of pairs of menus given to student s and her submitted ranking.

Say that the sequence of menus offered and actions chosen for a student s up to history h are as follows:

$$((\psi^1, a^1), (\psi^2, a^2), \dots, (\psi^t, a^t))$$

For simplicity, and without any loss of generality, assume that the sequence above has removed from the list the pairs $(\varnothing, \diamondsuit)$. We show below that menus given to students never include contracts present in any ranking submitted in previous steps.

Claim. If contract x is in a^t, then for every t' such that $t' > t$, $x \notin \psi^{t'}(s)$.

Proof. Let $c = c(x)$. If $\psi^{t'}(s) = \{\diamondsuit\}$, the claim obviously holds. Therefore, we consider the case in which both $\psi^t(s)$ and $\psi^{t'}(s)$ have a positive number of contracts available. Since x is in a^t, $x \in \psi^t(s)$. Also, since $x \in a^t$ and the fact that $\psi^{t'}(s) \neq \{\diamondsuit\}$, it must be the case that all the contracts in a^t are in $A^{t'-1}(c)$ (otherwise the GIDAM mechanism would still use the ranking a^t in step t'). Also, by the definition of the GIDAM, $\nexists y \in X_s, c \in C : y \in f_c\left(A^{t'-1}(c)\right)$, and in particular $x \notin f_c\left(A^{t'-1}(c)\right)$. Therefore, $x \notin \psi^{t'}(s)$. $\square$

Therefore, there is no repetition of contracts in a^i, $i = 1, \dots, t$. We will abuse notation and use a^i to represent the student's choice both as a ranking and as a set of contracts. Denote $\psi^i_- \equiv \psi^i \backslash \bigcup_{j=i}^t a^j$ and $X_s^+ \equiv X_s \backslash \{\varnothing\}$. We will show that this sequence could have been generated by a straightforward strategy of a student with a preference relation in the following class of preferences:[27]

$$X_s^+ \backslash \psi^1 \ R_s^* \ a^1 \ P_s^* \ \psi^1_- \backslash \psi^2 \ R_s^* \ a^2 \ P_s^* \ \psi^2_- \backslash \psi^3 \ R_s^* \cdots R_s^* \ a^t \ P_s^* \ \psi^t_-$$

The notation above includes a class of strict preferences because some of its elements $(X_s^+ \backslash \psi^1, \ \psi^1_- \backslash \psi^2, \ \text{etc})$ consist of (possibly empty) sets of contracts. Any strict preference

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[27]Note that this class of preferences does not necessarily include all the preferences that are compatible with the choices made.

derived from some ordering over the elements of each of those sets belongs to the class of preferences that we are referring to. We will refer by P_s^* to some arbitrary element of those preferences. The claim below implies that each preference in that class is complete over the set of contracts and that no contract appears more than once.

Claim. $\psi_-^t \subseteq \psi_-^{t-1} \subseteq \cdots \subseteq \psi_-^1 \subseteq X_s$, and $a^i \cap \psi_-^j = \varnothing$ for all i, j.

Proof. First, note that $\psi_-^1 = \psi^1 \backslash \bigcup_{j=1}^t a^j$. Since by definition ψ^1 is nonempty, $a^1 \subseteq \psi^1$, and since by definition $\psi^1 \subseteq X_s$, it follows that $\psi_-^1 \subsetneq X_s$. By the definition of ψ^t and Lemma 1, $\psi^k \subsetneq \psi^{k-1}$. Therefore:

$$\psi^{k-1} \backslash \bigcup_{j=k-1}^t a^j = \left((\psi^{k-1} \backslash \psi^k) \cup \psi^k \right) \backslash \left(a^{k-1} \cup \bigcup_{j=k}^t a^j \right)$$

Consider now any $k > 1$. By definition, all contracts in a^{k-1} are in ψ^{k-1}, and by the first claim in this proof, no contract in a^{k-1} is in ψ^k. Therefore:

$$\psi_-^{k-1} = \psi^{k-1} \backslash \bigcup_{j=k-1}^t a^j = \left(\left((\psi^{k-1} \backslash \psi^k) \backslash a^{k-1} \right) \cup \psi^k \right) \backslash \bigcup_{j=k}^t a^j = \psi_-^k \cup \left((\psi^{k-1} \backslash \psi^k) \backslash a^{k-1} \right)$$

That is, $\psi_-^{k-1} = \psi_-^k \cup \left((\psi^{k-1} \backslash \psi^k) \backslash a^{k-1} \right)$, which implies that $\psi_-^k \subseteq \psi_-^{k-1}$. Finally, for every $j \geqslant i$, it comes from the definition of ψ_-^i that $a^j \cap \psi_-^i = \varnothing$. Suppose instead that there is a $i > j$ such that $a^j \cap \psi_-^i = I$, for some non-empty set of contracts I. In that case, the definition of ψ_-^i implies that $I \subseteq \psi^i$. But in that case, we have that the contracts in I were submitted in a ranking by the student in step j and was available in the menu in step $j > i$, which contradicts the first claim in the proof.

Now, take some of the menus that were offered, ψ^i. We now show that for all $a \in \psi^i$ where $a \notin a^i$, $a^i P_s^* a$. For that, it suffices to show that:

$$a \in \bigcup_{j=i+1}^t a_j \cup \bigcup_{j=i}^{t-1} \psi_-^j \backslash \psi_-^{j+1} \cup \psi_-^t$$

That is, we will show that a must be at some element to the right of those in a^i in the definition of P_s^*. Since $a \notin a^i$, this is equivalent to:

$$a \in \bigcup_{j=i}^t a_j \cup \bigcup_{j=i}^{t-1} \psi_-^j \backslash \psi_-^{j+1} \cup \psi_-^t$$

Since we defined $\psi_-^i \equiv \psi^i \backslash \bigcup_{j=i}^t a^j$, we can rewrite the condition as:

$$a \in \underbrace{\psi^i \backslash \psi_-^i}_{(i)} \cup \underbrace{\bigcup_{j=i}^{t-1} \psi_-^j \backslash \psi_-^{j+1}}_{(ii)} \cup \underbrace{\psi_-^t}_{(iii)}$$

Suppose not. Then a cannot be in (i), (ii) and (iii). By (i), it must be that $a \notin \psi^i \backslash \psi_-^i$. Since $a \in \psi^i$, that implies $a \in \psi_-^i$. By (ii), since $a \notin \psi_-^i \backslash \psi_-^{i+1}$, it must then be that $a \in \psi_-^{i+1}$. This reasoning can be repeated until finding that it must be that $a \in \psi_-^t$. But that is (iii), which leads to a contradiction. $\square$

The sequence $\left(\left(\psi^1, a^1 \right), \left(\psi^2, a^2 \right), \dots, \left(\psi^t, a^t \right) \right)$ is consistent with student s having a preference over contracts P_s^* and following a straightforward strategy that, up in each step $k \leqslant t$ submits a ranking with the top $\left| a^k \right|$ contracts, among those available, with respect to her preference.

This implies that, since all other students follow straightforward strategies, every deviating strategy for student s is outcome-equivalent to following a straightforward strategy for some preference over contracts that is not necessarily that student's real preference P_s. Proposition 1, therefore, shows that the outcome produced will be the student-optimal stable matching with respect to the preference profile (P_s^*, P_{-s}). Theorem 7 in Hatfield and Kojima [2010] shows that since colleges' choice functions satisfy unilateral substitutes and the law of aggregate demand, submitting a true ranking is a dominant strategy when using a direct mechanism. In light of the result above, when other students follow straightforward strategies, any deviating strategy for s is outcome-equivalent to a deviating strategy in the direct mechanism, and as a result is not profitable.

We have two more steps to follow. One is to consider the fact that the definition of OPBE implies that no deviation is profitable *starting from any information set*. Obviously, any deviation may be outcome-relevant when starting from an information set in which a student receives a non-empty menu. Let t be the first step in which student s is given a non-empty menu while using the deviating strategy. It is easy to see that the continuation game is equivalent to one in which all contracts that were rejected at some step before t are removed from X.[28] Therefore, everything said above also holds when starting from that step.

The last step is to show that deviating strategies are stochastically dominated by straightforward ones under this equilibrium. Since we focus on pure strategies, the only source of uncertainty is the draw of P and X that takes place in history h_0. The fact that a truthful

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[28] This will not change the continuation choices by colleges due to the fact that they satisfy IRC.

ranking is a dominant strategy of the direct mechanism that yields the student-optimal stable matching implies that, regardless of other students' preferences (and the set of contracts X), the outcome that a student obtains by using the true preference is always weakly better than any other strategy. In particular, this implies that following any straightforward strategy will give her her most preferred contract whenever there exists a strategy that yields that while the realization of other students' preferences makes it possible. Also, due to the strategy dominance in the direct mechanism, straightforward strategies will always match a student with the second most preferred contract whenever X and other students' preferences are such that the first is not possible and the second is for some strategies. This can be done for every contract in the student's preference, and proves that any straightforward strategy stochastically dominates any deviating strategy. $\qquad\square$

Proposition 5

Proof. By the end of each step t, a mass $M^t = 1 - \zeta^*_{c_t}$ of students, who have grades in the range $\left[\zeta^*_{c_t}, 1\right]$ are permanently matched to some college. This is the case because by the end of step t all colleges $c_1, \ldots, c_t$ have their cutoffs equal to their final values. The expression for M^t can be found in the proof of Theorem 3 in Ashlagi and Shi [2014]:

$$M^t = (t-1)\, q + \frac{\sum_{j=t}^m r_j}{r_t} q$$

At each step t, for every college in $\{c_t, \ldots, c_m\}$, a positive mass of students with grades in the range $\left[0, \zeta^*_{c_t}\right]$, applies to that college, there will also be those who have grades that are above the final cutoff for that college. Therefore, the expression above is a strict lower-bound on the mass of students who are permanently matched. Consider now the expression $M^{t+1} - M^t$:

$$M^{t+1} - M^t = \left(\frac{q}{r_{t+1}} - \frac{q}{r_t}\right) \sum_{j=t+1}^m r_j =$$

$$= q\left(\frac{1}{r_{t+1}} - \frac{1}{r_t}\right) \sum_{j=t+1}^m r_j$$

If we open the values of r_i, we get:

$$q\left(e^{-\alpha v_{t+1}} - e^{-\alpha v_t}\right)\left(e^{\alpha v_{t+1}} + e^{\alpha v_{t+2}} + \ldots + e^{\alpha v_m}\right) =$$

$$= q \left(e^0 + e^{\alpha(v_{t+2} - v_{t+1})} + e^{\alpha(v_{t+3} - v_{t+1})} + \cdots + e^{\alpha(v_m - v_{t+1})} - e^{\alpha(v_{t+1} - v_t)} - e^{\alpha(v_{t+2} - v_t)} - \cdots - e^{\alpha(v_m - v_t)} \right)$$

Replacing the values of v_t with $(m - t)$, we get:

$$q \left(e^0 + e^{-\alpha} + e^{-2\alpha} + \cdots + e^{-\alpha(m-t-1)} - e^{-\alpha} - e^{-2\alpha} - \cdots - e^{-\alpha(m-t)} \right)$$

Therefore, the expression for $M^{t+1} - M^t$ is:

$$M^{t+1} - M^t = q \left(1 - e^{-\alpha(m-t)} \right)$$

Notice that $\frac{\partial \left(M^{t+1} - M^t \right)}{\partial \alpha} > 0$. Finally, the expression for those permanently matched according to the above-mentioned reasoning in the first step is:

$$M^1 = \frac{\sum_{j=1}^m r_j}{r_1} q = \frac{e^{-\alpha(m-1)} \left(e^{m\alpha} - 1 \right)}{e^\alpha - 1} q$$

So:

$$\frac{\partial M^1}{\partial \alpha} = -q \frac{e^{\alpha - (m-1)} \left(m - 1 + e^{m\alpha} - m e^\alpha \right)}{\left(e^\alpha - 1 \right)^2}$$

Since $m > 1$, $e^{m\alpha} > m e^\alpha$ and $\frac{\partial M^1}{\partial \alpha} < 0$. $\qquad \square$

A.5 Additional simulation data

A.5.1 Number of steps in IDAM vs maximum rank in DA when market balance is 5.0
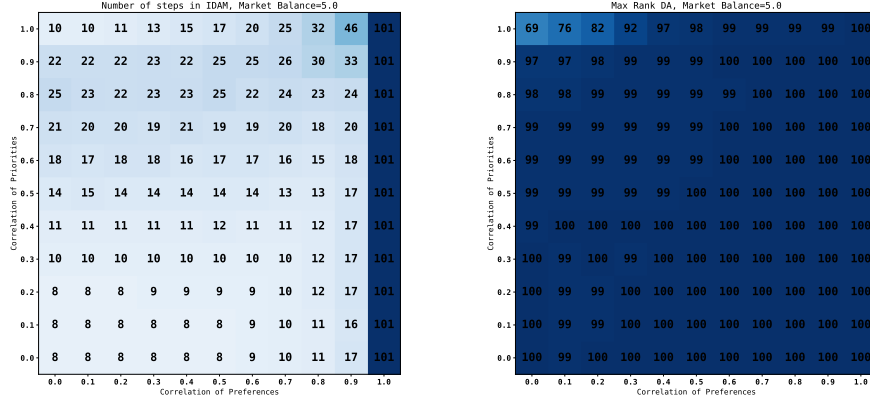
Figure A.1: Number of steps in IDAM vs maximum rank in DA

A.5.2 Detailed results for simulations of the IDAM+DA mechanism

		$IDAM(0+\infty)$			$IDAM(1+\infty)$			$IDAM(2+\infty)$			$IDAM(3+\infty)$		
	λ	0.5	1.0	1.5	0.5	1.0	1.5	0.5	1.0	1.5	0.5	1.0	1.5
α													
0.0		2.1	12.8	100.0	0.45	8.9	69.3	0.1	7.45	49.7	0.0	6.6	35.6
0.1		2.35	17.75	100.0	0.45	12.55	70.9	0.2	10.45	51.45	0.0	9.45	36.25
0.2		3.3	29.6	100.0	1.2	20.8	73.8	0.85	16.85	52.7	0.25	14.55	38.0
0.3		5.1	55.25	100.0	2.3	41.3	77.3	1.35	33.05	56.8	1.0	27.6	41.8
0.4		7.95	78.75	99.95	4.45	62.8	82.1	3.15	49.45	61.65	2.4	41.8	45.35
0.5		13.4	95.4	99.95	8.4	82.75	86.8	5.6	69.3	69.05	3.8	59.2	53.3
0.6		20.4	98.5	100.0	14.85	89.15	91.35	9.6	77.35	76.7	6.5	66.65	61.7
0.7		29.6	99.2	100.0	24.85	92.65	94.0	18.55	83.45	83.85	13.35	74.15	71.25
0.8		38.25	99.85	100.0	35.0	95.8	96.2	29.95	90.25	90.25	25.15	82.65	82.05
0.9		45.6	99.95	100.0	43.25	97.45	97.35	40.55	94.45	94.55	37.1	90.45	90.2
0.10		50.0	100.0	100.0	48.0	98.0	98.0	47.0	97.0	97.0	46.0	96.0	96.0

Table 1: Simulations of $IDAM(k+\infty)$ results when $\beta = 0.0$ and $0 \leqslant k \leqslant 3$

	λ	IDAM(4 + ∞)			IDAM(5 + ∞)			IDAM(6 + ∞)			IDAM(7 + ∞)		
		0.5	1.0	1.5	0.5	1.0	1.5	0.5	1.0	1.5	0.5	1.0	1.5
α													
0.0		0.0	6.1	25.45	0.0	5.65	18.85	0.0	5.4	13.25	0.0	5.0	9.55
0.1		0.0	8.65	25.95	0.0	7.65	19.1	0.0	7.0	13.1	0.0	6.85	9.6
0.2		0.0	12.8	26.6	0.0	11.65	19.0	0.0	10.85	12.9	0.0	10.0	9.2
0.3		0.55	23.7	29.9	0.25	21.0	21.05	0.15	19.1	14.5	0.1	17.8	10.1
0.4		1.5	36.15	33.7	1.15	30.9	23.65	0.95	27.2	16.5	0.7	24.2	11.05
0.5		2.6	50.2	40.2	2.0	43.95	29.75	1.6	38.2	20.65	1.3	33.4	13.85
0.6		4.6	57.15	48.25	3.35	49.75	36.7	2.9	43.4	27.55	1.95	38.0	19.1
0.7		9.95	64.75	59.25	6.6	56.3	47.65	4.9	49.4	37.1	3.7	43.55	28.1
0.8		19.65	74.75	72.6	15.45	66.8	62.5	12.05	59.1	53.15	9.15	52.7	43.95
0.9		33.8	86.25	85.35	29.6	80.55	79.05	25.7	74.95	73.15	22.35	69.1	65.95
0.10		45.0	95.0	95.0	44.0	94.0	94.0	43.0	93.0	93.0	42.0	92.0	92.0

Table 2: Simulations of $IDAM(k + \infty)$ results when $\beta = 0.0$ and $4 \leqslant k \leqslant 7$

	λ	IDAM(8 + ∞)			IDAM(9 + ∞)			IDAM(10 + ∞)		
		0.5	1.0	1.5	0.5	1.0	1.5	0.5	1.0	1.5
α										
0.0		0.0	4.8	7.05	0.0	4.5	5.1	0.0	4.3	3.4
0.1		0.0	6.4	7.0	0.0	6.3	5.15	0.0	6.05	3.45
0.2		0.0	9.2	6.85	0.0	8.5	4.9	0.0	8.15	3.3
0.3		0.05	16.8	7.05	0.05	15.95	5.05	0.0	14.85	3.2
0.4		0.55	22.1	7.4	0.3	19.9	4.7	0.15	18.45	3.0
0.5		1.05	29.4	9.1	0.8	26.65	5.45	0.55	24.55	3.45
0.6		1.7	33.45	12.4	1.3	29.6	7.95	1.15	26.5	4.15
0.7		2.95	38.45	20.5	2.2	33.2	13.6	1.75	29.1	8.3
0.8		6.8	46.55	34.55	5.1	41.1	27.1	3.9	36.05	20.3
0.9		18.75	63.75	59.35	15.4	58.0	52.6	13.05	52.9	45.85
0.10		41.0	91.0	91.0	40.0	90.0	90.0	39.0	89.0	89.0

Table 3: Simulations of $IDAM(k + \infty)$ results when $\beta = 0.0$ and $8 \leqslant k \leqslant 10$

	λ	IDAM(0 + ∞)			IDAM(1 + ∞)			IDAM(2 + ∞)			IDAM(3 + ∞)		
		0.5	1.0	1.5	0.5	1.0	1.5	0.5	1.0	1.5	0.5	1.0	1.5
α													
0.0		2.3	60.3	66.35	0.45	25.45	29.0	0.05	13.6	14.95	0.0	7.05	8.1
0.1		2.9	67.9	72.65	0.65	30.1	33.65	0.0	15.85	18.2	0.0	9.3	9.65
0.2		4.0	89.05	82.85	1.1	46.55	44.0	0.35	25.85	24.2	0.05	15.0	14.3
0.3		6.0	91.7	89.05	1.75	55.7	54.2	0.95	33.5	32.65	0.45	20.8	19.85
0.4		9.75	97.35	95.0	3.65	68.7	67.65	1.95	44.35	44.3	0.95	29.4	28.65
0.5		17.25	97.85	98.1	9.25	76.75	78.7	4.15	54.15	57.6	2.4	38.1	40.55
0.6		24.0	98.95	99.45	15.2	83.9	87.05	8.05	66.35	69.4	4.45	50.85	54.2
0.7		33.05	99.65	99.5	25.9	90.25	91.45	17.6	76.95	79.2	11.1	63.85	66.65
0.8		40.3	99.9	99.9	36.05	94.25	95.3	29.95	87.0	88.75	23.35	77.95	79.9
0.9		46.45	99.95	99.95	43.9	96.85	97.05	40.45	93.05	93.8	36.65	89.05	89.5
0.10		50.0	100.0	100.0	48.0	98.0	98.0	47.0	97.0	97.0	46.0	96.0	96.0

Table 4: Simulations of $IDAM(k + \infty)$ results when $\beta = 1.0$ and $0 \leqslant k \leqslant 3$

	λ	IDAM(4 + ∞)			IDAM(5 + ∞)			IDAM(6 + ∞)			IDAM(7 + ∞)		
		0.5	1.0	1.5	0.5	1.0	1.5	0.5	1.0	1.5	0.5	1.0	1.5
α													
0.0		0.0	4.25	4.7	0.0	2.5	2.6	0.0	1.7	1.7	0.0	1.05	0.85
0.1		0.0	6.1	5.7	0.0	3.45	3.45	0.0	2.35	2.1	0.0	1.3	1.5
0.2		0.0	9.05	8.4	0.0	5.25	5.1	0.0	3.25	3.4	0.0	1.9	2.5
0.3		0.0	13.2	12.25	0.0	7.95	7.65	0.0	4.8	5.35	0.0	2.85	3.3
0.4		0.75	19.85	19.25	0.2	13.2	13.3	0.1	8.6	8.75	0.05	5.95	6.0
0.5		1.35	27.6	28.25	0.55	19.1	20.55	0.15	13.55	15.3	0.05	9.55	10.3
0.6		2.65	38.55	41.55	1.55	29.85	31.0	1.1	22.6	23.55	0.9	16.95	18.6
0.7		7.15	52.8	55.15	4.6	42.4	44.75	2.85	35.25	36.2	1.8	28.6	29.15
0.8		17.55	68.65	70.8	12.65	59.8	61.65	9.25	52.35	54.3	6.75	45.05	47.05
0.9		33.05	83.85	84.6	28.9	78.7	78.9	24.75	73.05	73.95	21.8	67.7	69.0
0.10		45.0	95.0	95.0	44.0	94.0	94.0	43.0	93.0	93.0	42.0	92.0	92.0

Table 5: Simulations of $IDAM(k + \infty)$ results when $\beta = 1.0$ and $4 \leqslant k \leqslant 7$

α	λ	$IDAM(8+\infty)$			$IDAM(9+\infty)$			$IDAM(10+\infty)$		
		0.5	1.0	1.5	0.5	1.0	1.5	0.5	1.0	1.5
0.0		0.0	0.65	0.45	0.0	0.25	0.25	0.0	0.15	0.0
0.1		0.0	0.95	0.9	0.0	0.45	0.3	0.0	0.2	0.1
0.2		0.0	1.3	1.5	0.0	0.65	0.7	0.0	0.3	0.45
0.3		0.0	1.95	2.45	0.0	1.5	1.7	0.0	0.8	1.25
0.4		0.05	4.3	4.1	0.0	2.9	2.95	0.0	1.85	1.95
0.5		0.05	6.35	7.1	0.0	4.65	5.05	0.0	3.45	3.55
0.6		0.3	12.95	13.2	0.1	9.2	10.0	0.05	6.9	7.75
0.7		1.3	22.6	23.6	1.05	18.9	19.4	0.65	14.8	15.5
0.8		4.65	39.5	40.4	3.35	33.8	34.95	2.4	29.4	29.65
0.9		18.25	62.25	63.75	15.25	57.25	58.95	12.65	52.95	54.2
0.10		41.0	91.0	91.0	40.0	90.0	90.0	39.0	89.0	89.0

Table 6: Simulations of $IDAM(k+\infty)$ results when $\beta = 1.0$ and $8 \leqslant k \leqslant 10$